②

AFOSR · TR · 8 9 - 0 2 1 7

FINAL REPORT

Development of Adaptive Grid Schemes
Based on Poisson Grid Generators

AFOSR - 85 -0195
UTA CFD Report No. 89-02

Principal Investigator
Dale A. Anderson

Major Staff
Stephen R. Kennon

D'

AD-A206 922

COLLEGE OF ENGINEERING

DTIC
SELECTED
FEB 1 6 1989
D

The University
of Texas at Arlington
Arlington, Texas

89 · 2 15 11

FINAL REPORT

Development of Adaptive Grid Schemes
Based on Poisson Grid Generators

AFOSR - 85 -0195
UTA CFD Report No. 89-02

Principal Investigator
Dale A. Anderson

Major Staff
Stephen R. Kennon

January 1989

DTIC
ELECTE
FEB 1 6 1989
D

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED		1b. RESTRICTIVE MARKINGS	
2a. SECURITY CLASSIFICATION AUTHORITY		3. DISTRIBUTION/AVAILABILITY OF REPORT	
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE		Approved for public release; distribution unlimited	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)		5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 89-0217	
6a. NAME OF PERFORMING ORGANIZATION University of Texas at Arlington	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION AFOSR/NA	
6c. ADDRESS (City, State, and ZIP Code) Arlington, Texas 76019		7b. ADDRESS (City, State, and ZIP Code) AFOSR/NA Bldg. 410 Bolling AFB, DC 20332-5260	
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR	8b. OFFICE SYMBOL (If applicable) NA	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-85-0195	

8c. ADDRESS (City, State, and ZIP Code) AFOSR/NA Bldg. 410 Bolling AFB, DC 20332-5260	10. SOURCE OF FUNDING NUMBERS			
	PROGRAM ELEMENT NO. 61102F	PROJECT NO. 2307	TASK NO A1	WORK UNIT ACCESSION NO

11. TITLE (Include Security Classification)
Development of Adaptive Grid Schemes Based on Poisson Grid Generators (U)

12. PERSONAL AUTHOR(S)
Anderson, Dale A.

13a. TYPE OF REPORT Final	13b. TIME COVERED FROM 85/5/15 TO 88/11/14	14. DATE OF REPORT (Year, Month, Day) 89-1-10	15. PAGE COUNT 49

16. SUPPLEMENTARY NOTATION
UTA CFD Report No. 89-02

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Grid Generation, Adaptive, Unstructured, Poisson Equation
	20,04		

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Poisson equation grid generators are the most popular differential equation mesh generation schemes. Results reported under this program show how to use the existing schemes to construct adaptive grids. Methods for controlling individual arc lengths as well as cell volumes are given and techniques for constructing two-dimensional orthogonal adaptive grids are included. Adaptivity was also demonstrated with unstructured meshes using the Poisson equation as a control on grid point location. Additional adaption schemes for unstructured grids are presented using a linear spring analogy.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☐ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED	
22a. NAME OF RESPONSIBLE INDIVIDUAL Dr. L. Sakell	22b. TELEPHONE (Include Area Code) (202) 767-4935	22c. OFFICE SYMBOL NA

DD Form 1473, JUN 86   *Previous editions are obsolete.*   SECURITY CLASSIFICATION OF THIS PAGE

Summary

Research on constructing adaptive grid generation schemes has been
supported for the past 42 months under AFOSR-85-0195. This research has
primarily centered on identifying ways of using the familiar Poisson grid
generators as adaptive grid schemes. The control of mesh point spacing
has been achieved by deriving the relationship between the arc length or
cell volumes and the grid control functions of the Poisson grid generators.
Applications of these schemes have been made to a variety of flow problems.
These concepts have been applied to the generation of both structured and
unstructured adaptive grids. Activities supported under this grant are
summarized and details of the status of current research are presented.

Introduction

The original grant AFOSR-85-0195 was targeted toward applying the concept of equidistribution of a weight function over arcs or cell volumes. Early in this program, it became apparent that application of strict equidistribution to produce adaptive grids was very difficult. Lack of smoothness as well as skewness created problems when the grid scheme was coupled with a numerical method for solving fluid flow problems. These difficulties persist in two and three dimensions and relegate the application of simple arc equidistribution to one dimension. Applications in more than one dimension produce generally unacceptable results.

Research under this program was redirected in order to avoid the shortcomings of arc equidistribution laws. The implementation of strict arc equidistribution as the control law in adaptive grid generation was abandoned and was used as a guide in evaluating the potential of other approaches. Using this philosophy, the standard Thompson grid generation scheme was written in a form that can be identified as a nonlinear equidistribution scheme. The relationship between the grid control functions of the Thompson scheme and the weight functions of equidistribution was analytically established providing a mechanism where the existing Thompson method can be directly used to generate adaptive grids. This development has been adequately reported in the literature and will not be repeated here.

The use of Thompson's method as a nonlinear equidistribution scheme permits control of differential arc lengths in each computational coordinate direction. This requires that one control or weight function be specified in each computational coordinate direction. It is advantageous to compute the solution for an adaptive grid where area in two dimensions or volume

in three dimensions is controlled. This requires that only one weight function be specified as opposed to two or three when arc elements are manipulated. A method where the grid control functions were directly related to cell volume was developed. This method is based on the original Winslow diffusion idea, one weight function may be specified and an adaptive grid controlling cell volume can be created. This work has also been reported in publications and lectures and details are omitted here.

The identification of the grid control functions with weight functions of arc equidistribution and the diffusion with cell area in the usual Poisson grid generation equations are, in the author's opinion, the most significant contribution to grid generation technology for a number of years. For a long time, grid point distributions were determined by using certain types of functions for grid control without any understanding of why arc lengths or cell volumes were influenced. The control of arcs and volumes through the grid control functions was developed analytically and verified by numerical experiment under this grant. This is a major breakthrough in constructing static grids and understanding point location and control function selection. Adaptive grids can now be computed because the cause and effect relationships have been established. From a practical point of view, most government laboratories and industrial organizations have Poisson grid generators in house. With nearly trivial code changes and almost no software development cost, the existing code can be modified and used as an adaptive grid scheme.

The Thompson/Winslow equations for generating grids form an elliptic system. Some effort was expended in evaluating the practicality of using the parabolic form of these equations to produce an adaptive grid. The

solution of the elliptic system is rather slow and imp _mentation of the

parabolic approximation ideally should be much more efficient. Some suc-

cess with this idea was achieved and the ideas have been published.

The Poisson equations can be used to generate arbitrary mesh systems

and the control of cell areas/volumes can be achieved using the equidis-

tribution-grid control relationships developed under this program. As

a result, effort has been expended in constructing both structured and un-

structured grids. This has been successful and results have been published

in the literature.

Research Status

The results of research are normally presented at conferences or published in archive journals as results are produced. Since this is a final report, a number of topics are incomplete and papers have not been submitted for publication. While the status of these efforts has been reported earlier, a brief statement noting the current disposition of the research is necessary.

1. Orthogonal Adaptive Grids.

Work on structured two-dimensional orthogonal mesh generators has been essentially complete for several months. It remains to submit the development for publication. The basic governing equations for generating such grids can be derived by discarding the cross derivative in Thompson's grid equations and writing

$$\alpha \; (\vec{r}_{\xi\xi} + \phi\vec{r}_{\xi}) + \gamma \; (\vec{r}_{\eta\eta} + \psi\vec{r}_{\eta}) = 0 \tag{1}$$

where

$$\vec{r} = (x,y)^T$$

$$\alpha = x_{\eta}^2 + y_{\eta}^2 = S_{\eta}^2$$

$$\gamma = x_{\xi}^2 + y_{\xi}^2 = S_{\xi}^2$$

In terms of the arc ratio for orthogonal grids

$$\vec{r} + \frac{f_{\xi}}{f} \vec{r}_{\xi} + \frac{1}{f^2} (\vec{r}_{\eta\eta} - \frac{f_{\eta}}{f} \vec{r}_{\eta}) = 0 \tag{2}$$

where S is arc length and

$$f = \frac{S_{\eta}}{S_{\xi}}$$

The adaptivity is controlled in this system through f. The role of the grid control functions is taken by f and the arc derivatives serve as the means of exercising adaptive control in the grid.

The arc derivatives are most easily established by using the concept of equidistribution. Let w_1 and w_2 be weight functions for S_ξ and S_η respectively. Thus

$$S_\xi w_1 = c_1$$

$$S_\eta w_2 = c_2$$

For simplicity, let c_1 and c_2 be unity. The arc ratio, f, may then be written

$$f = \frac{w_1}{w_2} \tag{3}$$

Both derivatives of f in the governing equation are logarithmic

$$\frac{\partial \log f}{\partial \xi} = \frac{1}{w_1} \frac{\partial w_1}{\partial \xi} - \frac{1}{w_2} \frac{\partial w_2}{\partial \xi} \tag{4}$$

This form for grid control in boty ξ and η leads to problems where w_1 and w_2 are nearly equal. In that case, adaption literally vanishes because the derivative of w_1 cancels that of w_2.

The form of the weight functions is usually

$$w_1 = 1 + A \left(\frac{\partial u}{\partial \xi}\right)^2 \tag{5}$$

Success has been achieved by letting A change as

$$A = 1 + b \left|\frac{\partial u}{\partial \xi}\right| \left|\frac{\partial u}{\partial \xi}\right| \tag{6}$$

where b is a constant. The effect of this form is to prevent the reduction of adaptivity in regions where $u_\xi^2 \cong u_\eta^2$. Application of these

developments will be included in a paper soon to be submitted for publication.

2. Linear Elliptic Adaptive Grid Generation

Several popular grid generation methods are based on a set of linear elliptic governing equations. However, these linear equations are transformed from physical (x,y)-space to computational (ξ,η)-space. The transformed equations then become non-linear, and are solved in computational space for the (x,y) coordinates of the grid points. Although the original equations are linear and un-coupled, the transformed equations are non-linear and strongly coupled. Therefore, the possibility of solving the original linear equations in physical space is attractive and could result in the development of a new adaptive grid generation method.

The basis for this new adaptive grid generation method is solving a set of linear elliptic partial differential equations in physical space. As such, it is necessary to have an initial, base grid on which the solution is to be found. This base grid can be generated by any method as long as it is reasonably smooth and non-over-lapped. Once the base grid is defined, grids can be generated that are smooth, adapted to specified weight functions, and orthogonal at specified boundaries (structured case). The basic elliptic equation set is that of the diffusion grid generation method.

The diffusion method of Anderson has proven to be a very power-ful adaptive grid generation method. The governing equations are given by the set

$$\nabla \cdot D\nabla\xi = 0 \qquad (1)$$

$$\nabla \cdot D\nabla \eta = 0$$

where $D(x,y)$ is the 'diffusion' function. The diffusion function D (x,y) can be shown to be directly related to the local cell area (i.e. the Jacobian) of the grid that results from solving the diffusion equation set. Therefore, if one wishes to have a grid that adapts to a positive weight function $W(x,y)$, then one should chose the diffusion function such that

$$DW = constant \tag{2}$$

Thus, where the weight function is large, the diffusion will be small, and in turn the local grid spacing will be small. There are several advantages to this method:

1. The method satisfies an extremum principle for any choice of $D(x,y) \geq 0$. This means that the method is guaranteed to produce grids without overlap.

2. Only one grid control function (namely D) need be chosen. This can be contrasted to other elliptic-based methods that require the specification of two grid control functions (e.g. the Thompson scheme requires the choice of two functions P and Q).

3. The diffusion equations can be transformed into equations that are identical to the Thompson equations where:

$$\nabla^2 \xi = P(x,y) = \frac{-\nabla D}{D} \cdot \nabla \xi$$

$$\nabla^2 \eta = Q(x,y) = \frac{-\nabla D}{D} \cdot \nabla \eta$$

The normal practice when solving elliptic grid generation equations is to interchange the dependent (ξ,η) and independent (x,y) variables to get an equation set of the form (for the diffusion method):

$$ax_{\xi\xi} - 2Bx_{\xi\eta} + \gamma x_{\eta\eta} = (ax_{\xi} - Bx_{\eta})\frac{D_{\xi}}{D} - (Bx_{\xi} - \gamma x_{\eta})\frac{D_{\eta}}{D}$$

$$ay_{\xi\xi} - 2By_{\xi\eta} + \gamma y_{\eta\eta} = (ay_{\xi} - By_{\eta})\frac{D_{\xi}}{D} - (By_{\xi} - \gamma y_{\eta})\frac{D_{\eta}}{D}$$

where

$$a = x_{\eta}^2 + y_{\eta}^2$$

$$B = x_{\xi}x_{\eta} + y_{\xi}y_{\eta}$$

$$\gamma = x_{\xi}^2 + y_{\xi}^2$$

One can see that these equations are non-linear and coupled due to the presence of the factors a, B, γ which depend on both x and y. Instead of solving this non-linear set of equations in computational space, it is clear that one can solve the linear uncoupled set in physical space. The advantages of this approach are:

1. Since the governing equation set is linear, elliptic and satisfies an extremum principle, there exists a unique solution.

2. Standard iterative methods (such as successive overrelaxation, conjugate gradient, and others) are guaranteed to converge regardless of the initial guess for the solution.

3. The possibility exists for using very fast direct solution techniques such as cyclic reduction and fast-fourier-transform schemes. Therefore, the adaptive computational grid may be computed in one direct-solution step, without iteration.

4. Implementation of Neumann boundary conditions becoms almost trivial. In many cases, it is desirable to have the grid lines be orthogonal at certain boundaries of the domain. This results in a Neumann boundary condition which can be imposed

4. directly on the linear grid equations. This is
   opposed to methods that solve the equations in computational
   space. In that case, the grid points must slide along the
   boundary and therefore the boundary must be spline-fit after
   each iteration. This incurs additional expense and program
   complexity.

5. No interpolation is needed during each iteration to update
   the values of the weight function at the grid points. In a
   standard method, the weight function must be updated after
   each iteration, which is a very costly procedure.

The real advantage of this method will be experienced in the gener-
ation of three-dimensional adaptive grids. Standard three-dimensional
adaptive grid generation methods require that the grid points slide
along boundaries. This requires a sophisticated surface definition
method (such as bi-cubic patches). On the other hand, the new linear
adaptive method does not require the movement of points along any
boundary.

The method was tested on a set of model problems set in the unit
square. The first example (Fig. 1) shows the result of adaption to
the function

$$W(x,y) = \begin{cases} 1 & \text{if } r < \frac{1}{4} \\ 2 & \text{if } r \geq \frac{1}{4} \end{cases} \tag{3}$$

where

$$r = (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2$$

The grid adapts well to the weighting function and is clearly uniform

where W is uniform. The second example (Fig. 2) involves adaption

to a model of a shock wave where the weighting function is given by

$W = 1 + 10|\nabla u|^2$ where

$$u = \tanh(2x - y) \qquad\qquad (4)$$

The grid adapts to the model shock wave in a reasonably smooth manner.

It should be noted that the grid was specified to be orthogonal to

all boundaries.

The linear adaptive grid generation method has also been tested

on realistic turbomachinery problems. In all cases it has proven to

be robust and efficient and should find wide application in computational

fluid dynamics.

3. Generation of Unstructured Triangular Grids Using Elliptic Partial

    Differential Equations.

The past few years have seen a growing interest in the use of un-

structured triangular grids in computational fluid dynamics. One of

the major difficulties in the generation of unstructured grids is to

produce grids that are smooth, non-overlapped and adaptive to a

specified weight function. Therefore, a new method has been devel-

oped for generating adaptive unstructured triangular grids based on

elliptic partial differential equations (PDE). The use of an el-

liptic PDE ensures that the resulting grid is non-overlapped and

smooth, while adaption is achieved by using Anderson's diffusion

method.

The new adaptive grid generation method is based on solving the

equations (see above discussion for more details on the diffusion method)

$$\nabla \cdot D\nabla\xi = 0$$

$$\nabla \cdot D\nabla\eta = 0 \tag{1}$$

These equations are transformed to computational space to give a

set of equations to be solved for the (x,y)-coordinates of the grid:

$$ax_{\xi\xi} - 2Bx_{\xi\eta} + \gamma x_{\eta\eta} = (ax_\xi - Bx_\eta)\frac{D_\xi}{D} - (Bx_\xi - \gamma x_\eta)\frac{D_\eta}{D}$$

$$ay_{\xi\xi} - 2By_{\xi\eta} + \gamma y_{\eta\eta} = (ay_\xi - By_\eta)\frac{D_\xi}{D} - (By_\xi - \gamma y_\eta)\frac{D_\eta}{D}$$

where

$$a = x_\eta^2 + y_\eta^2$$

$$B = x_\xi x_\eta + y_\xi y_\eta$$

$$\gamma = x_\xi^2 + y_\xi^2$$

For an unstructured grid, the computational space would also be unstruc-

tured (as opposed to a structured grid that produces a uniform grid in

computational space). Therefore, instead of transforming to computa-

tional space, the concept of a master element is employed. Each

triangular element in physical space is mapped to the single triangular

master element. All computations are done on the master element and

then mapped back to physical space. The governing equations are dis-

cretized using a standard finite element procedure and solved by simple

point relaxation. The method was tested on a model problem set in the

unit circle. The initial unstructured grid is shown in Fig. 3 and was

generated using a Delaunay triangulation procedure. Figure 4 shows

the result of requiring that the grid adapt to the weight function

$$W(x,y) = \begin{cases} 10 & \text{if } r < \tfrac{1}{4} \\ \dfrac{1}{10} & \text{if } r \geq \tfrac{1}{4} \end{cases} \tag{2}$$

where

$$r = (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2$$

Note that where the weight function is large, the grid cells are small, and vice versa. Also note that the grid is relatively uniform where the weight function is uniform.

This research has shown that methods that have been successfully used to generate structured adaptive grids can be applied to generate unstructured adaptive grids. The new unstructured grid adaption method should find wide applicability in the new flow solvers being developed for triangular grids.

References:

1. Anderson, D.A., "Adaptive Grid Scheme Controlling Cell Area/ Volume," AIAA Paper No. 87-0202, Reno, Nevada, January 1987.

2. Thompson, J.F., F.C. Thames, and C.M. Mastin, "Automatic Numerical Generation of Body Fitted Curvilinear Coordinate Systems for Fields Containing any Number of Arbitrary Two-Dimensional Bodies," Journal of Computational Physics, Vol. 15, 1974, pp. 299-319.

3. Winslow, A., "Numerical Solution of the Quasi-Linear Poisson Equation," Journal of Computational Physics, Vol. 1, 1966, pp 149-172.

4. Jameson, A., T.J. Baker and N.P. Weatherill, "Calculation of Inviscid Transonic Flow over a Complete Aircraft," AIAA paper 86-0103, Reno, Nevada, January 1986.

5. Lohner, R., et al, "Finite Element Methods for High Speed Flows," AIAA paper 85-1531, Cincinnati, Ohio, July 1985.

6. Becker, E.B., G.F. Carey, and J.T. Oden, "Finite Elements: An Introduction," Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1981.

References: (continued)

7. Kennon, S.R., "Numerical Solution of Weak Forms of Conservation Laws on Optimal Unstructured Triangular Grids," Ph.D. Thesis, Department of Aerospace Engineering and Engineering Mechanics, University of Texas at Austin, 1987.
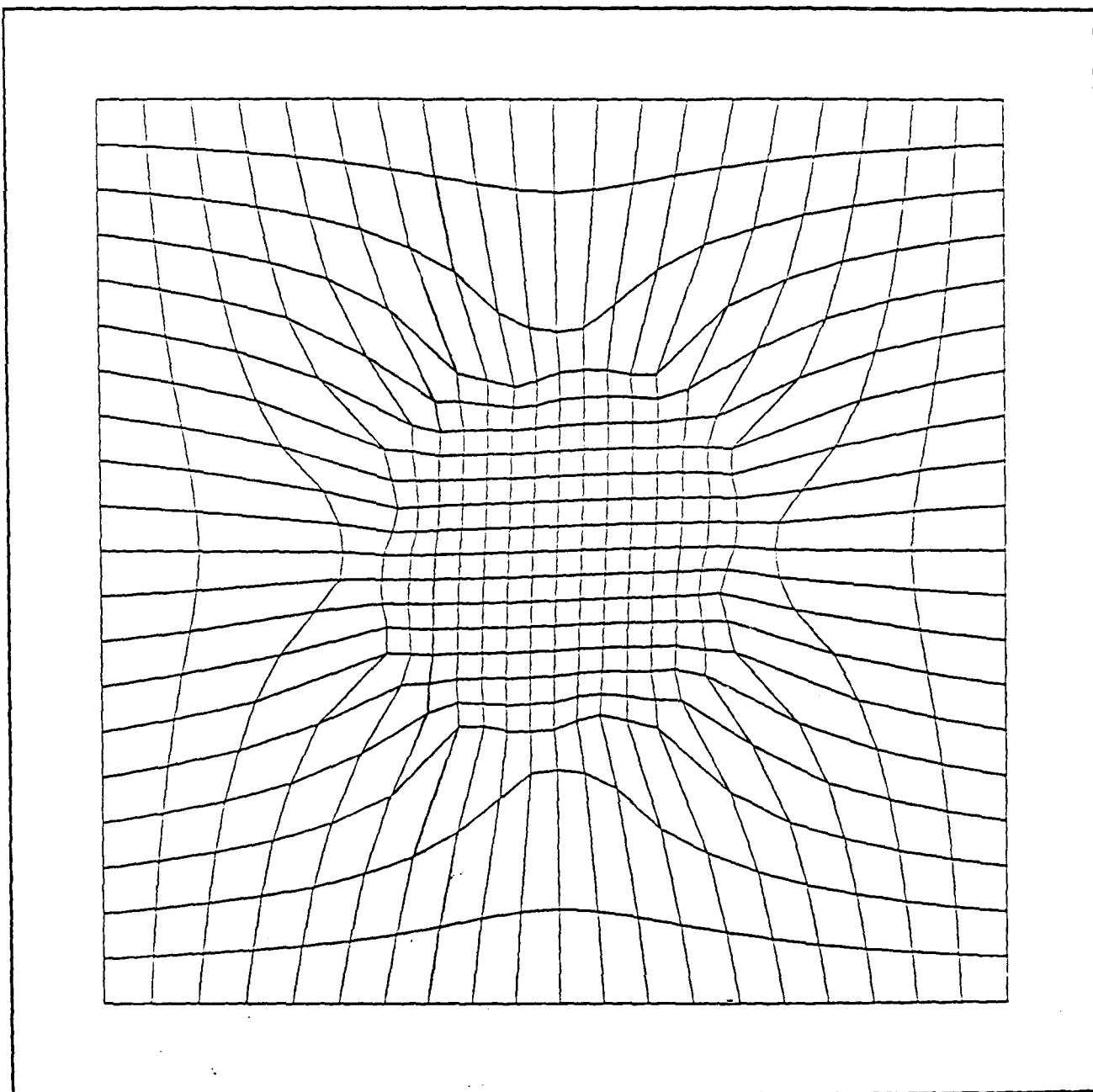
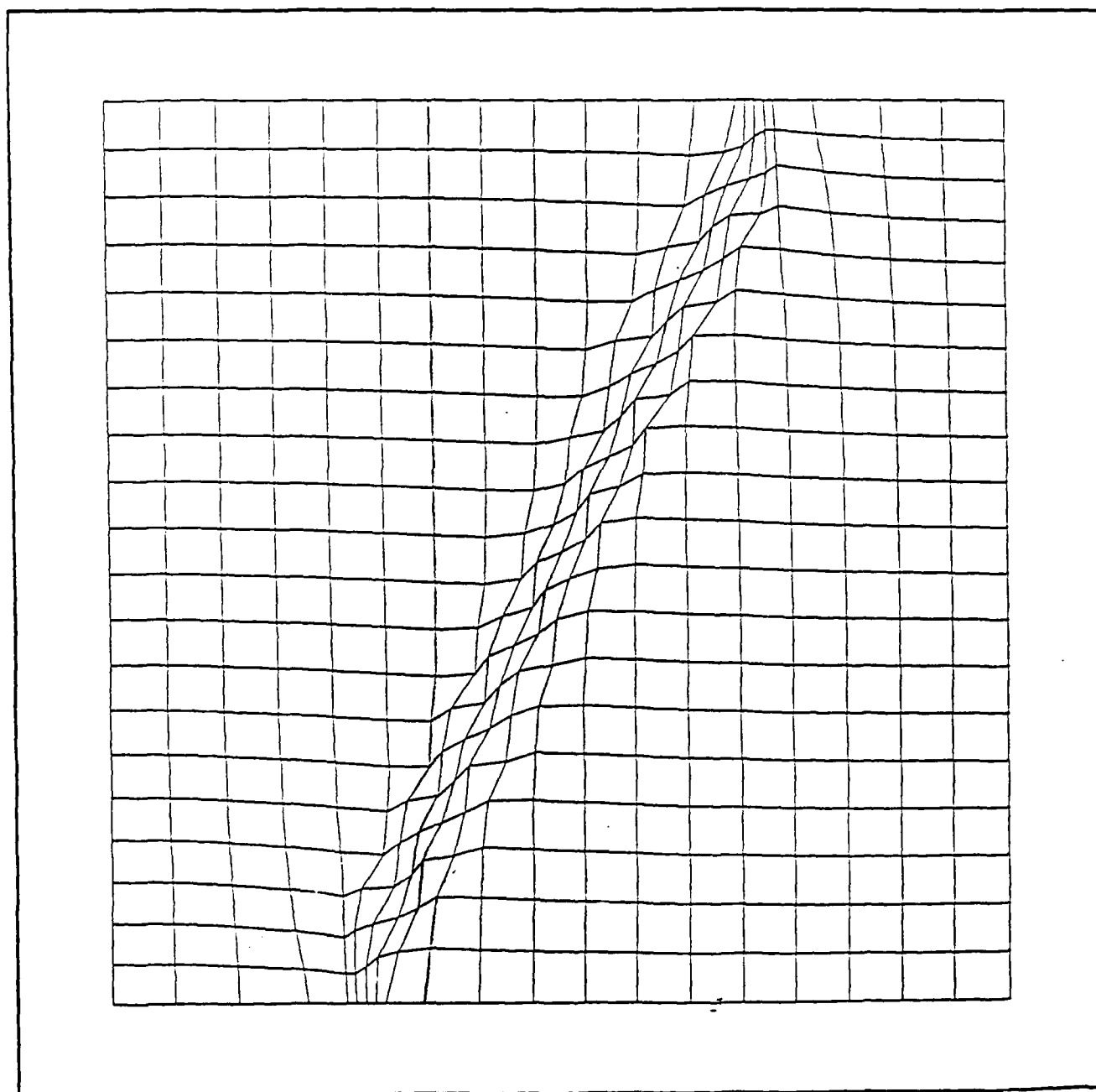Figure 1   Linear adaptive grid generation-Example 1
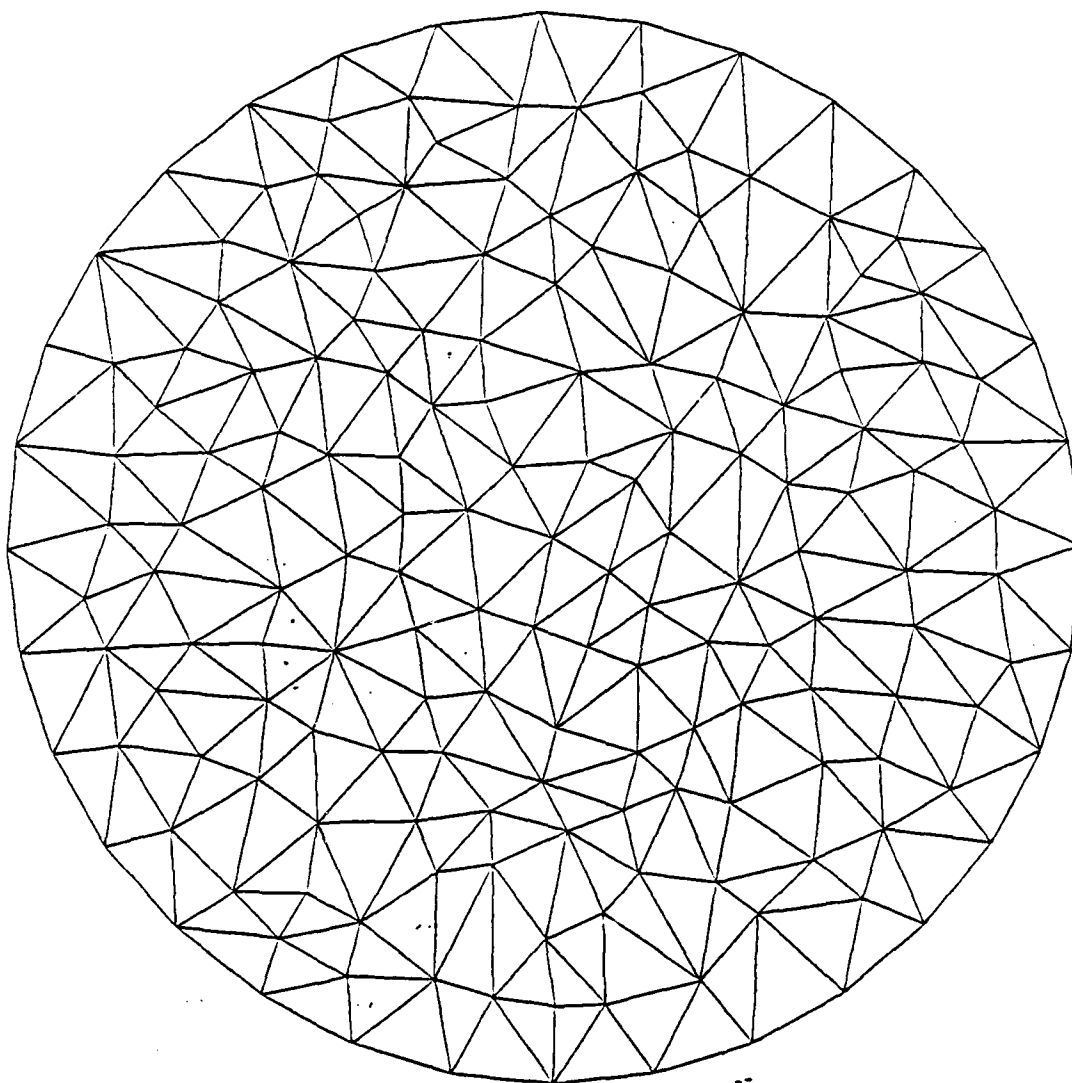
Figure 2   Linear adaptive grid generation-Example 2

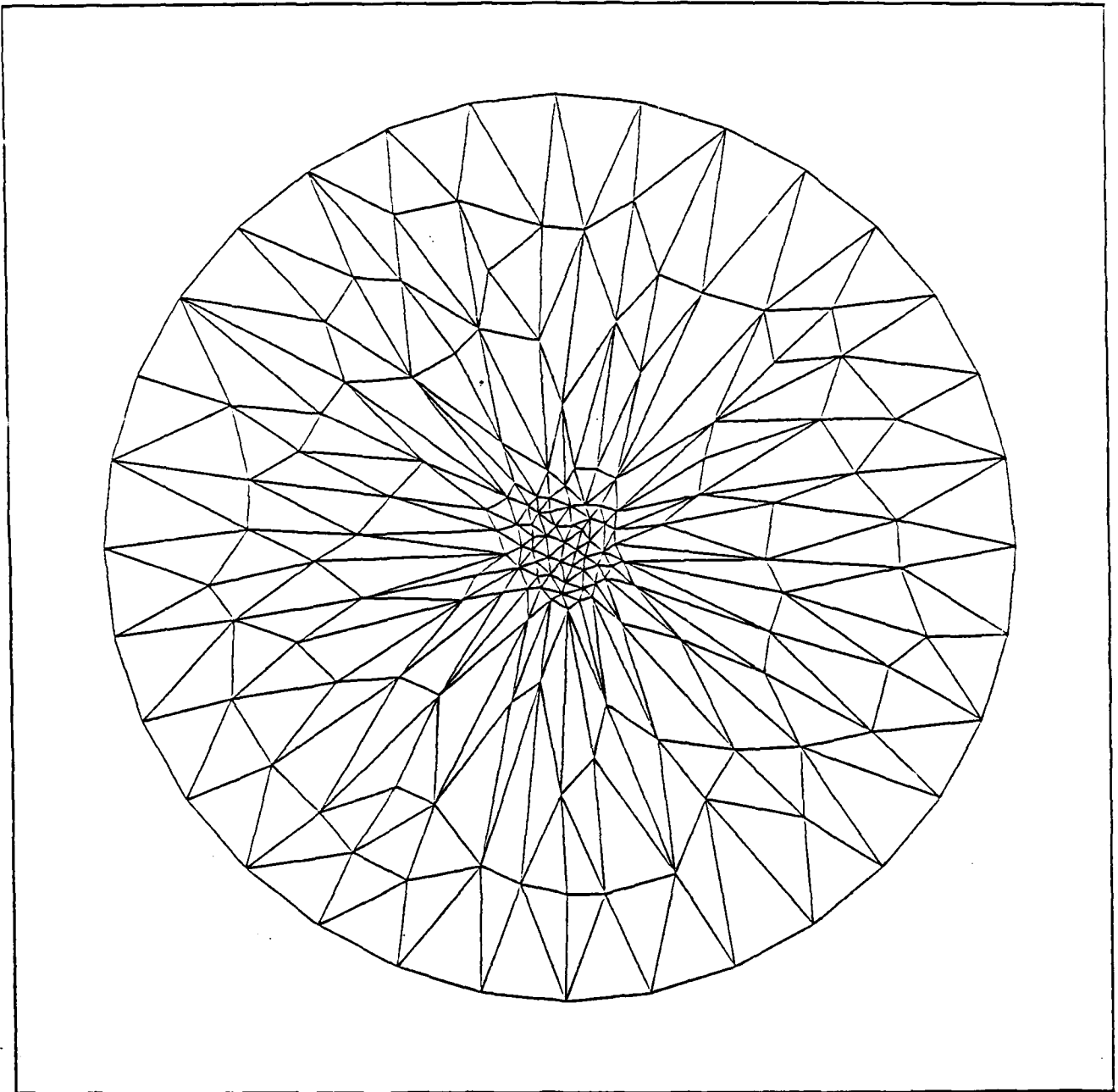Figure 3 Unstructured Adaptive Grid Generation: Initial Grid

Figure 4   Unstructured Adaptive Grid Generation: Adapted Grid

Conclusions

This program has been a successful one where major constibutions to the construction of adaptive grid schemes have been made. These contributions were made in applying Poisson grid generation schemes as solution adaptive mesh generators. While application of these concepts in a dynamic environment is possible, the major impact at present is in using the adaptive Poisson generators in static applications.

Present techniques for dynamic adaption are based on integration of the equations governing the physical problem of interest and then remeshing and refining the meshes a separate process. Better ways of generating dynamically adaptive grids must be developed. For example, if the steady grid law is differentialed to obtain the grid speeds, the grid and the equations of motion can be integrated in time to any order and no interpolations of the dependent variables is necessary. Research is being conducted in this area and should be continued in order to better resolve the issues in dynamic adaption for both structured and unstructured grids.

Publications Resulting from Research Supported Under this Grant

Anderson, Dale A., "Application of Poisson Grid Generators to Problems in Fluid Dynamics." Invited lecture to be presented at the 7th International Conference on Finite Element Methods in Flow Problems, Huntsville, AL, April 1989.

Anderson, Dale A., "Grid Cell Volume Control with an Adaptive Grid Generator." Submitted for publication in Applied Mathematics and Computation.

Parpia, I. and Noack, R., "Solution Adaptive Parabolic Grid Generation in Two and Three Dimensions." Submitted for presentation at the Second International Conference on Numerical Grid Generation in Computational Fluid Mechanics, Miami, December 1988.

Noack, Ralph W. and Anderson, Dale A., "Solution Adaptive Grid Generation Using Parabolic Differential Equations," AIAA Paper 88-0315, January 1988.

Thompson, David S. and Anderson, Dale A., "A Psuedo-Unsteady Approach for Predicting Steady Supersonic Flows," Presented at the AIAA 25th Aerospace Sciences Meeting, Reno, Nevada, AIAA Paper 87-0541, January 1987. Also accepted for publication in the AIAA Journal.

Anderson, Dale A., "An Adaptive Grid Scheme Controlling Cell Area/Volume," Presented at the 25th Aerospace Sciences Meeting, Reno, Nevada, AIAA Paper 87-0102, January 1987.

Anderson, Dale A., "Constructing Adaptive Grids with Poisson Grid Generators," Proceedings of the First International Conference on Numerical Grid Generation in Computation Fluid Dynamics, Landshut, W. Germany, July 1986.

Anderson, Dale A., "Equidistribution Schemes, Poisson Generators and Adaptive Grids," Applied Mathematics and Computation, Vol. 24, No. 3, Part 1, Dec. 1987, pp. 211-227.

Publications (continued)

Anderson, Dale A. and Steinbrenner, J. P., "Generating Adaptive Grids with a Conventional Grid Scheme," AIAA Paper 86-0427, January 1986.

Graduate Students Supported Under this Program

Ross, John - M.S. in Aerospace Engineering

    "Application of an Adaptive Grid Scheme to a Shock Capturing Code"

Bishop, David - M.S. thesis in preparation

Noack, Ralph - Ph.D. candidate

Professional Personnel

Principal Investigator

    Dale A. Anderson

Major Staff

    Stephen R. Kennon

Appendix

The three papers in this section are included in order to register the major contributions of this research.

1) Anderson, Dale A., "Application of Poisson Grid Generators to Problems in Fluid Dynamics." Invited lecture to be presented at the 7th International Conference on Finite Element Methods in Flow Problems, Huntsville, AL, April 1989.

2) Anderson, Dale A., "Grid Cell Volume Control with an Adaptive Grid Generator." Submitted for publication in Applied Mathematics and Computation.

3) Anderson, Dale A., "Equidistribution Schemes, Poisson Generators and Adaptive Grids," Applied Mathematics and Computation, Vol. 24, No. 3, Part 1, December 1987, pp. 211-227.

APPLICATION OF POISSON GRID GENERATORS TO PROBLEMS IN FLUID DYNAMICS

by

Dale A. Anderson
Aerospace Engineering Department
University of Texas at Arlington
Arlington, Texas 76019

Abstract

Two schemes for constructing solution adaptive grids are reviewed in this paper. The first method is designed to control arc length elements comprising the computational cell boundaries while the second directly controls cell area/volume. These techniques are both applied using the popular Poisson grid generation equations by identifying the grid control functions in terms of the physical solution. A number of examples are presented showing grids produced with these methods. Adaptive grid capability can be added to existing Poisson grid generators with very little additional investment.

1. Introduction

The development of techniques for constructing solution adaptive grids has received significant emphasis over the past few years. The problem addressed in this discussion of adaptive grids is to determine the distribution of a fixed number of grid points on a physical domain in order to improve the quality of the numerical solution. In this approach, the grid points are moved during the calculations and the point locations as well as the vector of dependent variables emerge as part of the solution. Solution quality can not be specifically defined since the exact objectives which dictate the numerical simulation are different in each case. Usually some measure of global error is minimized. However, in many cases, simply improving resolution in regions of rapidly changing dependent variable is satisfactory.

Many schemes have been put forward for generating adaptive grids. Early work was based on the concept of the equidistribution of weight function to control arc length and this idea worked well in one dimension [1, 2]. In two dimensions the idea of equidistribution produced grids without any smoothness measure and for high adaption rates, severe grid distortion was evident [3,4,5]. The distortion issue was addressed in the work of Nakahashi and Diewert [6] where an angularity control was incorporated in the formulation for controlling rate of change of inclination of cell sides.

A more elegant formulation of the adaptive grid problem was provided by Brackbill and Saltzman [7]. In this approach, a functional containing contributions from orthogonality, smoothness and adaption was minimized. The Euler-Lagrange equations produced by this formulation form an elliptic system for the calculation of the grid point locations in physical space. The advantage to this approach is that the relative importance of each measure in grid construction can be controlled. The concept has been used in constructing grids for a number of interesting examples in fluid mechanics. The disadvantage is that the method is complex in the sense that the grid equations are complicated and require large computational resources in two dimensions. The formulation in three dimensions appears to be impractical for the same reasons.

In more recent work, Anderson [8,9] and Anderson and Steinbrenner [10] have devised a way of using the Poisson grid generation equations [11,12] to construct adaptive grids. The grid control functions are related to weight functions using the concept of equidistribution providing a way of directly constructing adaptive grids with very little change to existing software. Two techniques have been developed. The first is based on control of arc elements and the second controls cell area/volume. These two formulations are reviewed in the following sections and a number of examples of the application of the ideas to fluid flow problems are presented.

2. The ARC Control Scheme

Perhaps the most widely used grid generation scheme based on partial dif-ferential equations is that based on the ideas originally proposed by Winslow [13,14] and extended and implemented by Thompson et al. [11]. The original Winslow scheme provides a system of equations for a two-dimensional case of the form

$$\nabla^2 \xi = 0 \quad , \quad \nabla^2 \eta = 0 \tag{1}$$

for the computational coordinates (ξ, η). These equations assure that the grid is the smoothest possible if the smoothness measure is

$$I = \int\!\!\int \left[(\nabla \xi)^2 + (\nabla \eta)^2 \right] dx \, dy \tag{2}$$

The Euler-Lagrange equations resulting from minimizing Eq (2) are the grid Eqns (1). Adding source terms or grid control functions, P and Q, to Eqns (1) pro-vides a way of controlling interior point placement other than through the boundary conditions. The grid equations become

$$\nabla^2 \xi = P \quad , \quad \nabla^2 \eta = Q \tag{3}$$

These equations are usually written with computational coordinates, (ξ, η), as independent variables yielding a nonlinear partial differential equation for (x,y) of the form

$$\alpha \vec{r}_{\xi\xi} - 2\beta \vec{r}_{\xi\eta} + \gamma \vec{r}_{\eta\eta} = -J^2 \left[P\vec{r}_\xi + Q\vec{r}_\eta \right] \tag{4}$$

where

$$\vec{r} = (x,y)^T \quad , \quad \alpha = x_\eta^2 + y_\eta^2 \tag{5a}$$

$$\beta = x_\xi x_\eta + y_\xi y_\eta \quad , \quad \gamma = x_\xi^2 + y_\xi^2 \tag{5b}$$

and the Jacobian of the mapping is defined by

$$J = x_\xi y_\eta - x_\eta y_\xi \tag{5c}$$

The major problem in using the Thompson scheme (Eq (4)), is selecting the

proper values of the grid control functions. In references 8, 9 and 10, the relationship of the grid control functions to the weight functions of arc equidistribution schemes is derived. In an arc equidistribution scheme for a constant η curve where arc length in physical space is denoted by S and ξ varies along the curve, we may write

$$S_{\xi} W = C_1 \tag{6}$$

where W is the weight function and C_1 is a constant. Elements of arc length, ΔS, are small where W is large and vice versa. If the Thompson Eq (4) is written along constant ξ and constant η curves, the resulting system of two equations can be identified as approximating arc equidistribution laws (Eq (6)) and consequently, the relationship between the weight functions and the grid control functions may be identified as

$$P = \frac{\alpha}{J^2} \frac{1}{W_1} \frac{\partial W_1}{\partial \xi} \tag{7}$$

$$Q = \frac{\gamma}{J^2} \frac{1}{W_2} \frac{\partial W_2}{\partial \eta} \tag{8}$$

In these expressions, W_1, W_2 are the weight functions controlling arc length along η = constant and ξ = constant curves respectively. In three dimensions a similar expression is obtained in the ζ direction. If the values of W_1, W_2 are coupled to the solution of the physical problem, the grid on the physical domain can be made to be adaptive and reflect the important features of the physical process.

3. The Area/Volume Control Scheme

In a 1981 report [14], Winslow suggested constructing grids by employing a diffusion law of the form

$$\nabla \cdot (D \nabla \xi) = 0 \quad , \quad \nabla \cdot (D \nabla \eta) = 0 \tag{9}$$

The computational coordinates, (ξ, η), are controlled by the diffusion, D, when these expressions are used as the appropriate grid generation set. In terms of the Thompson grid control functions

$$\nabla^2 \xi = P = - \frac{\nabla \xi \cdot \nabla D}{D} \quad , \quad \nabla^2 \eta = Q = - \frac{\nabla \eta \cdot \nabla D}{D} \tag{10}$$

Since both P and Q are expressed in terms of a single diffusion term controlling the production of the computational coordinates, an obvious question is to ask of the D variable can be in any way related to cell area in two dimensions or volume in three dimensions. If the relationship between D and cell volume can be derived, then the grid control functions can be used to produce an adaptive grid by adjusting cell volume using a weight function approach.[9]

If the Laplacian operator is applied to the inverse Jacobian

$$I = \xi_x \eta_y - \eta_x \xi_y \tag{11}$$

and it is assumed that the quantity

$$\eta_{xy}(\xi_{xx} - \xi_{yy}) - \xi_{xy}(\eta_{xx} - \eta_{yy})$$

is negligible, it may be shown that the Jacobian, J, is related to the diffusion D by the expression

$$J = CD \tag{12}$$

where C is constant. In terms of the Thompson grid control functions

$$P = -\frac{\alpha}{J^2}\frac{D_\xi}{D} + \frac{\beta}{J^2}\frac{D_\eta}{D} \quad , \quad Q = \frac{\beta}{J^2}\frac{D_\xi}{D} - \frac{\gamma}{J^2}\frac{D_\eta}{D} \tag{13}$$

The actual adaption can be accomplished by assuming that

$$DW = \text{constant} \tag{14}$$

where W is a weight function related to the solution. Again, where W is large, D is small (cell area) and vice versa.

4. Examples and Conclusion

It is worthwhile to show some examples of grids computed using the ideas presented above. Figure 1 shows the grid produced using Eqns 4, 7 and 8 for supersonic flow through a compressor. The inlet Mach no. was 2.85 and the grid was adapted using a weight function dependent on the pressure gradient. The Euler equations were solved and the adapted grid shows the presence of the bow shock as well as the passage shock between the blades. Obviously, this example was computed with a very course grid.

Figure 2 shows the grid used to compute solutions of the Euler equations for Mach 10 flow over a six degree sharp cone with drag flaps [15] in a plane normal to the cone axis. The form of the weight function was selected to depend both on first derivature and curvature of the pressure along computational coordinate lines. The nearly conical shock produced by the vehicle is clearly visible in the clustering between the body and the outer boundary and the strong shock produced by the drag flap is clearly visible. Since this is a solution based on pressure information in an inviscid flow, no other flow details would influence the grid point locations.

Figure 3 shows the grid point distribution employed to solve the Euler equations for a three dimensional wing (23012). The root section is nearest the viewer and the tip is into the figure. Not all grid lines for this transonic calculation (M = .8, α = 2°) are shown. The grid was adapted using pressure gradient information with Eqns (7,8). The shock on the upper surface is clearly visible and clustering toward the wing tip shows althougn the perspective does not provide a very good view of this.

As a final example, Figure 4 shows the grid used to compute a solution

of the Navier-Stokes equations for normal injection in a supersonic stream. The equations for laminar flow were used and the weight function was assumed to depend on the pressure gradient and the velocity gradient. The shocks and shear layers for this example are viable in the grid. In the interests of brevity, the contour plots of flow variables have been omitted but the appropriate references can be consulted.

Both the arc control and cell volume control methods work well within the constraints imposed by the Poisson grid generation equations. The techniques are easily implemented in existing codes for a minimum investment of time. Either procedure may be used to rapidly gain an adaptive grid capability.

ACKNOWLEDGMENT

REFERENCES

1. Dwyer, H. A., Smooke, M. O., and Kee, R. J., AIAA J. 18:1205-1212 (1980).
2. Dwyer, H. A., Smooke, M. O., and Kee, R. J., Numerical Grid Generation, J. F. Thompson Ed., North Holland, 339-350 (1982).
3. Gnoffo, P. A., A Vectorized, Finite-Volume, Adaptive Grid Algorithm Applied to Planetary Entry Problems, AIAA paper 82-1018.
4. Anderson, D. A., Advances in Grid Generation, ASME, 1-15 (1983)
5. Eiseman, P., Alternating Direction Adaptive Grid Generation, AIAA paper 83-1937.
6. Nakahaski, K. and Deiwert, G. S., AIAA J. 24:945-954 (1986).
7. Brackbill, J. and Saltzman, J., Journal of Computational Physics, Vol.46, 342-368 (1982).
8. Anderson, D. A., Applied Mathematics and Computation, Vol 24, No. 3, Part 1, 211-227 (1987)
9. Anderson, D. A., An Adaptive Grid Scheme Controlling Cell Area/Volume, AIAA paper 87-0102 (1987).
10. Anderson, D. A. and Steinbrenner, J., Generating Adaptive Grids with a Conventional Grid Scheme, AIAA paper 86-0427.
11. Thompson, J. F., Thames, F. C., and Mastin, C. M., Journal of Computational Physics, Vol. 15, 299-319 (1974).
12. Thompson, J. F., Warsi, Z. U. A., and Mastin, C. W., Numerical Grid Generation, North-Holland (1985).
13. Winslow, A., Journal of Computational Physics 1:149-172 (1966).
14. Winslow, A., Adaptive Mesh Zoning by the Equipotential Method, UCID-19062 (1981).
14. Noack, R. W. and Anderson, D. A., Solution Adaptive Grid Generation Using Parabolic Differential Equations, AIAA paper 88-0315 (1988).
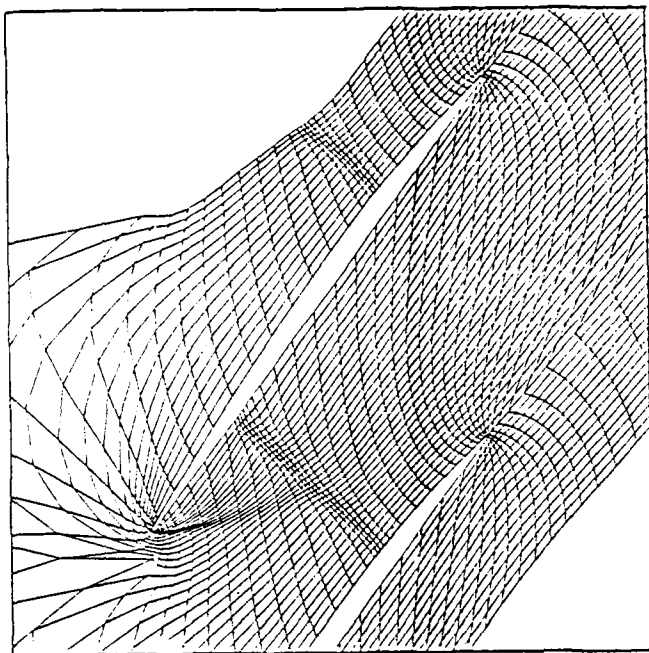
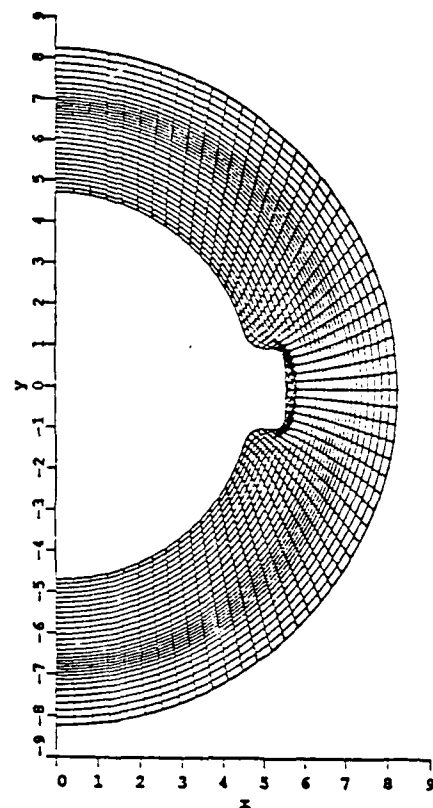Figure 1: Grid for Flow in Supersonic Compressor

Figure 2: Adaptive Grid for
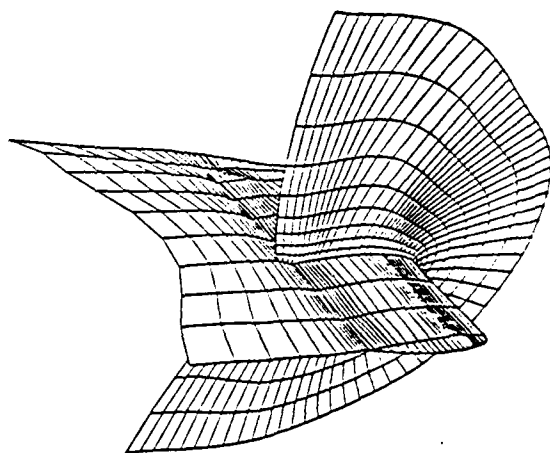Flapped Re-entry Vehicle

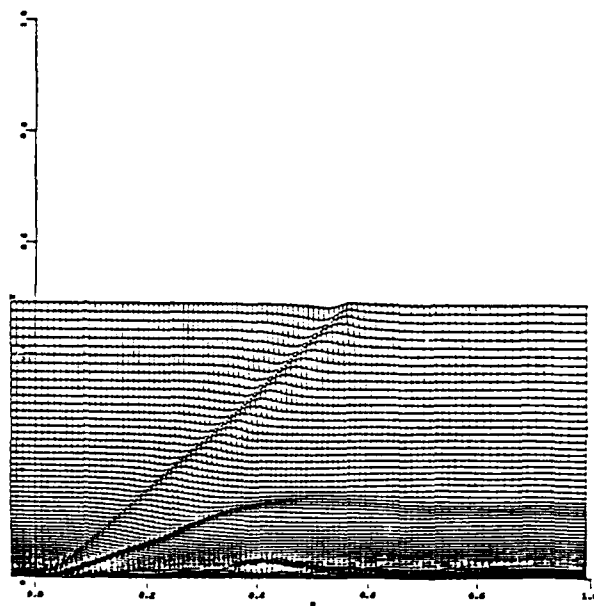Figure 3: Grid for Transonic Flow
over a wing

Figure 4: Grid for Supersonic Viscous
Flow over a Flat Plate with Normal
Injection

Grid Cell Volume Control with
an Adaptive Grid Generator

Dale A. Anderson

Department of Aerospace Engineering
University of Texas
Arlington, Texas 76019

ABSTRACT

A technique for generating solution adaptive grids controlling cell
volume is presented. The grid control functions of the popular Poisson
grid generators are defined to control cell size. An analytic derivation
showing the relationship of the grid control functions and the cell vol-
ume is presented and several examples are given illustrating the idea.
Only one function must be prescribed to control cell size when this tech-
nique is implemented in a Poisson grid generator.

INTRODUCTION

One of the first steps in applying a numerical method to solve a
partial differential equation is to construct a suitable mesh or grid
for the physical domain. There are many techniques for doing this but
the most popular method is that of Thompson et al (1). This scheme
employs the original method proposed by Winslow (2) but includes addi-
tional control over placement of interior grid points. This has led to
the popular Poisson mesh generators for constructing grids for arbitrar-
ily shaped regions.

Recently, a significant interest has surfaced in generating adaptive
grids. When an adaptive method is used, the mesh point locations change
while the solution of the physical problem is computed. Both the grid
point locations and the original physical unknowns evolve as part of the
problem solution. The grid points are moved during the calculation in
order to improve the quality of the result. This is usually assumed to
mean that some measure of total error in the solution has been reduced
or physical resolution has been improved in regions where large changes
in dependent variables occur. The issue of improved quality is problem
dependent and additional research will be required to adequately define
both grid and solution quality.

Several schemes for generating adaptive grids have been developed
using a variety of approaches. These include strict equidistribution
schemes employed by Dwyer et al (3), Gnoffo (4) and Rai and Anderson (5).
A modification of the strict arc equidistribution scheme to reduce grid
skewness was used by Nakahashi and Deiwert (6) and has been applied to
a number of problems.

A number of researchers have proposed mesh schemes based on deriving the grid generation equations by minimizing some grid function. Most notable of these methods is that of Brackbill and Saltzman (7) where a combination of contributions from smoothness, orthogonality and adaption were minimized. The Euler Lagrange equations that result provide the elliptic grid generating set for calculations of the grid point locations. As an adaptive mesh scheme, this technique can be applied and has been shown to work well in some applications. The expense of solving these equations along with the complexity of the system (especially in three dimensions) has prevented this technique from being widely used.

More recently, Anderson (8) and Anderson and Steinbrenner (9) have used the Thompson scheme as an adaptive grid generator. The grid control functions are related to the solution of the physical problem resulting in a direct way of controlling point locations in physical space. As the physical solution evolves, the grid can be recomputed providing grid point placement where desired. This represents a significant advance because very little software modification is required to make existing Poisson grid generation codes fully adaptive.

In the following sections, a technique for creating adaptive grids controlling cell volume is presented. This method is developed to work in conjunction with the Thompson scheme. The relationship between cell volume and the grid control functions is based on solution of a differential equation relating cell volume and the computational coordinates. A number of simple examples are given illustrating the application of the method in two dimensions.

A CELL VOLUME ADAPTIVE GRID SCHEME

The modification of the Thompson scheme to produce adaptive grids in Refs (8) and (9) works well and is attractive due to its simplicity. However, the grid control functions are modified to produce grids where arc equidistribution laws are approximately obeyed. Consequently, a weight function is required for each dimension. In three dimensions, these weight functions are required to provide the grid adaption. It may be advantageous to prescribe only one control or weight function to adjust the cell volume rather than three functions to independently adjust arc lengths.

A technique for producing a grid generation scheme controlling cell volume where only one function must be specified is possible. Winslow (2) suggested writing the grid generation equations in the form

$$\nabla \cdot (D \nabla \xi) = 0 \tag{1a}$$

$$\nabla \cdot (D \nabla \eta) = 0 \tag{1b}$$

where (ξ, η) are the computational coordinates and D is a diffusion. The diffusion controls the flux of computational coordinate through the surface of an arbitrary volume. If Eq (1a) is integrated throughout an arbitrary volume and the divergence theorem is applied,

$$\iint_S D\nabla\xi\cdot\vec{N}dA = 0 \tag{2}$$

From this expression, it is clear that D does indeed control the production of ξ (or η) in the volume. This may also be seen by expanding Eqs (1a) and (1b) to obtain the more familiar form

$$\nabla^2\xi = -\nabla\xi\cdot\frac{\nabla D}{D} \tag{3a}$$

$$\nabla^2\eta = -\nabla\eta\cdot\frac{\nabla D}{D} \tag{3b}$$

The right sides of Eqs (3a) and (3b) are recognized as the grid control functions in Thompson's method

$$P = -\nabla\xi\cdot\frac{\nabla D}{D} \tag{4a}$$

$$Q = -\nabla\eta\cdot\frac{\nabla D}{D} \tag{4b}$$

In earlier adaptive grid schemes based on Poisson grid generators (8), P and Q are each explicitly related to the changes in arc length in the ξ and η directions respectively. In the Winslow diffusion formulation, the P and Q control functions both depend on a single function, D, which determines the point distribution.

The diffusion term in the Winslow formulation can be directly related to the cell area or volume. Consider the Jacobian of the mapping defined as

$$I = \xi_x\eta_y - \xi_y\eta_x = \frac{\partial(\xi,\eta)}{\partial(x,y)} \tag{5}$$

for the two-dimensional case. Since the diffusion is related to $\nabla^2\xi$ and $\nabla^2\eta$, a reasonable step is to form $\nabla^2 I$.

$$\nabla^2 I = \nabla^2[\xi_x\eta_y - \xi_y\eta_x] \tag{6}$$

Expending, an expression of the form

$$\nabla^2 I = I\nabla^2\bar{D} - \nabla\bar{D}\cdot\nabla I + R \tag{7}$$

is obtained where R is defined as

$$R = 2\left[\nabla\left(\frac{\partial\xi}{\partial x}\right)\cdot\nabla\left(\frac{\partial\eta}{\partial y}\right) - \nabla\left(\frac{\partial\xi}{\partial y}\right)\cdot\nabla\left(\frac{\partial\eta}{\partial x}\right)\right] \tag{8}$$

and $\bar{D} = \ln D$. Introduce the inverse Jacobian

$$J = \frac{1}{I} \qquad (9)$$

and let

$$\bar{J} = \ell n J$$

This reduces Eq (7) to

$$\nabla^2 (\bar{J} - \bar{D}) - \nabla \bar{J} \cdot \nabla(\bar{J} - \bar{D}) = - JR \qquad (10)$$

The term on the right side, $-JR$, is assumed to be small by comparison with terms on the left. For those cases where this assumption is valid

$$\nabla^2 (\bar{J} - \bar{D}) - \nabla \bar{J} \cdot \nabla(\bar{J} - \bar{D}) = 0 \qquad (10a)$$

and a solution may readily be written as

$$\nabla(\bar{J} - \bar{D}) = 0$$

or

$$\bar{J} - \bar{D} = \text{Constant}$$

and $\quad D = C \; J \qquad (11)$

where C is a constant. The diffusion term in Winslow's scheme is simply the local cell area or volume scaled by a constant. With this result, the original grid control functions may be written in terms of the diffusion.

The original grid generation equations with (x,y) as independent variables are now written

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = [\alpha x_\xi - \beta x_\eta] \frac{D_\xi}{D} - [\beta x_\xi - \gamma x_\eta]\frac{D_\eta}{D} \qquad (12a)$$

$$\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = [\alpha y_\xi - \beta y_\eta] \frac{D_\xi}{D_\eta} - [\beta y_\xi - \gamma y_\eta]\frac{D_\eta}{D} \qquad (12b)$$

where

$$\alpha = x_\eta^2 + y_\eta^2$$

$$\beta = x_\xi x_\eta + y_\xi y_\eta$$

$$\gamma = x_\xi^2 + y_\xi^2$$

The local cell area may be controlled by prescribing the diffusion, D,

where it appears on the right side of Eqs (12a) and (12b). If an adaptive grid generation technique is desired, the diffusion or local area may be related to the computed dependent variables of the physical problem.

EXAMPLES

To illustrate the control of cell area with this diffusion formulation of the Poisson grid generation equations, a few examples are necessary. Suppose an analytic function u(x,y) is prescribed on the unit square. The grid point distribution can be controlled by using an area equidistribution law. Define a weight function W(x,y) so that

$$DW = constant \tag{13}$$

When the weight function is large, the diffusion (area) is small and vice versa. For the simple analytic examples presented here, it is sufficient to relate the weight function to the gradient of u(x,y).

$$W(x,y) = 1 + A(\nabla u)^2 \tag{14}$$

In this expression, A is a constant although it may be a prescribed function, where the degree of adaption is adjusted by increasing or decreasing A.

The first example is based on adapting a 31 x 31 point mesh to the pillbox function

$$u = 1. \qquad 0 \leq r \leq .15$$

$$u = 1.-5r \qquad .15 < r < .35$$

$$u = 0. \qquad .35 \leq r$$

The results of computing a solution for the grid point locations for this pillbox function using the weight functions given in Eq (14) are shown in Fig 1. Since the weight function only increases in regions where the gradient is nonzero, point clustering occurs in the region where .15<r<.35. In the regions where the function is constant, the grid is relatively uniform. The boundary points are allowed to move along constant x or y lines to adjust to the adaption provided through the grid control functions. On the boundary, it has been assumed that the second derivative of the floating coordinate with respect to the opposite family computational coordinate is zero. For example, along x = 0, y is computed on a ξ = constant surface. Thus, $y_{\xi\xi}$ is assumed to vanish along this boundary.

Fig 2 shows the results of applying the grid area adaption scheme to a problem of a simple shock-like function. The prescribed function is 0 to the left, one to the right and linearly transitions from 0 to 1 over 15% of the physical domain length. This example again employs the weight function of Eq (14) and illustrates the clustering of grid points

in regions of nonzero gradients.

The grid control functions of the Thompson scheme can be expressed in terms of Winslow's diffusion and adaptive grids controlling cell area/volume can be directly generated as evidenced by the examples shown. In Ref (8), arc length control was employed to adjust lengths and create a similar adaption. In cases where a single computational coordinate can be made to nearly align along an adaption surface, it is best to employ arc adaption rather than attempt to use the volume method. However, in problems where the form of the solution is unknown, the area/volume technique has the advantage of only requiring specification of one control function through the diffusion term.

The linear relationship between the Jacobian and the diffusion given by Eq (11) is approximate. The term on the right side of Eq (10) has been assumed to be small and has been neglected in this analysis. The meaning of this term is difficult to establish and the effect on the diffusion area relationship is hard to assess. The examples presented indicate that the influence is small. However, there may be cases where a more careful evaluation of the impact of this term is necessary. Any approximate analysis must be employed keeping the restrictions of the method in mind.

SUMMARY AND CONCULSIONS

The diffusion formulation of Winslow has been reviewed as a Poisson grid generation technique. The diffusion was shown to approximately be proportional to local cell volume or area. The grid control functions of the standard Poisson grid generators were related to the diffusion and several examples illustrating control of area were presented.

This technique is another way of using standard Poisson grid generators for creating adaptive grids. Either area/volume or arc length along constant computational coordinate surfaces may be controlled. The selection process depends on the particular application and the skill of the user in employing appropriate weight functions providing grid control. Like the arc adaption scheme, this volume/area control method can be implemented with only small changes in existing Poisson grid generators. The addition of this option provides users with an additional choice in constructing useful grids.

ACKNOWLEDGMENT

REFERENCES

1   J. F. Thompson, F. D. Thames, and C. W. Mastin, Automatic numerical
    generation of body-fitted curvilinear coordinate system for fields
    containing any number of arbitrary two-dimensional bodies. J. Comput.
    Phys. 15:299-319 (1974).

2   A. Winslow, Numerical solution of the quasi-linear Poisson equation,
    J. Comput. Phys. 1:149-172 (1966).

3   H. A. Dwyer, M. O. Smooke, and R. J. Kee, Adaptive grid method for
    problems in fluid mechanics and heat transfer, AIAA J. 18:1205-1212
    (1980).

4   P. A. Gnoffo, A Vectorized, Finite-Volume, Adaptive Grid Algorithm
    Applied to Planetary Entry Problems, AIAA paper 82-1018.

5   M. M. Rai and D. A. Anderson, Grid evolution in time asymptotic prob-
    lems, J. Comput. Phys. 43:327-344 (1981).

6   K. Nakahaski and G. S. Deiwert, Three-dimensional adaptive grid method,
    AIAA J. 24:945-954 (1986).

7   J. Brackbill and J. Saltzman, Adaptive zoning for singular problems
    in two dimensions, J. Comput Phys. 46:342-368 (1982).

8   D. A. Anderson, Equidistribution Schemes, Poisson Generators, and
    Adaptive Grids, Applied Mathematics and Computation 24:211-227 (1987).

9   D. A. Anderson and J. Steinbrenner, Generating Adaptive Grids with a
    Conventional Grid Scheme, AIAA paper 86-0427.

FIGURE CAPTIONS

Fig. 1 Grid for Pill Box

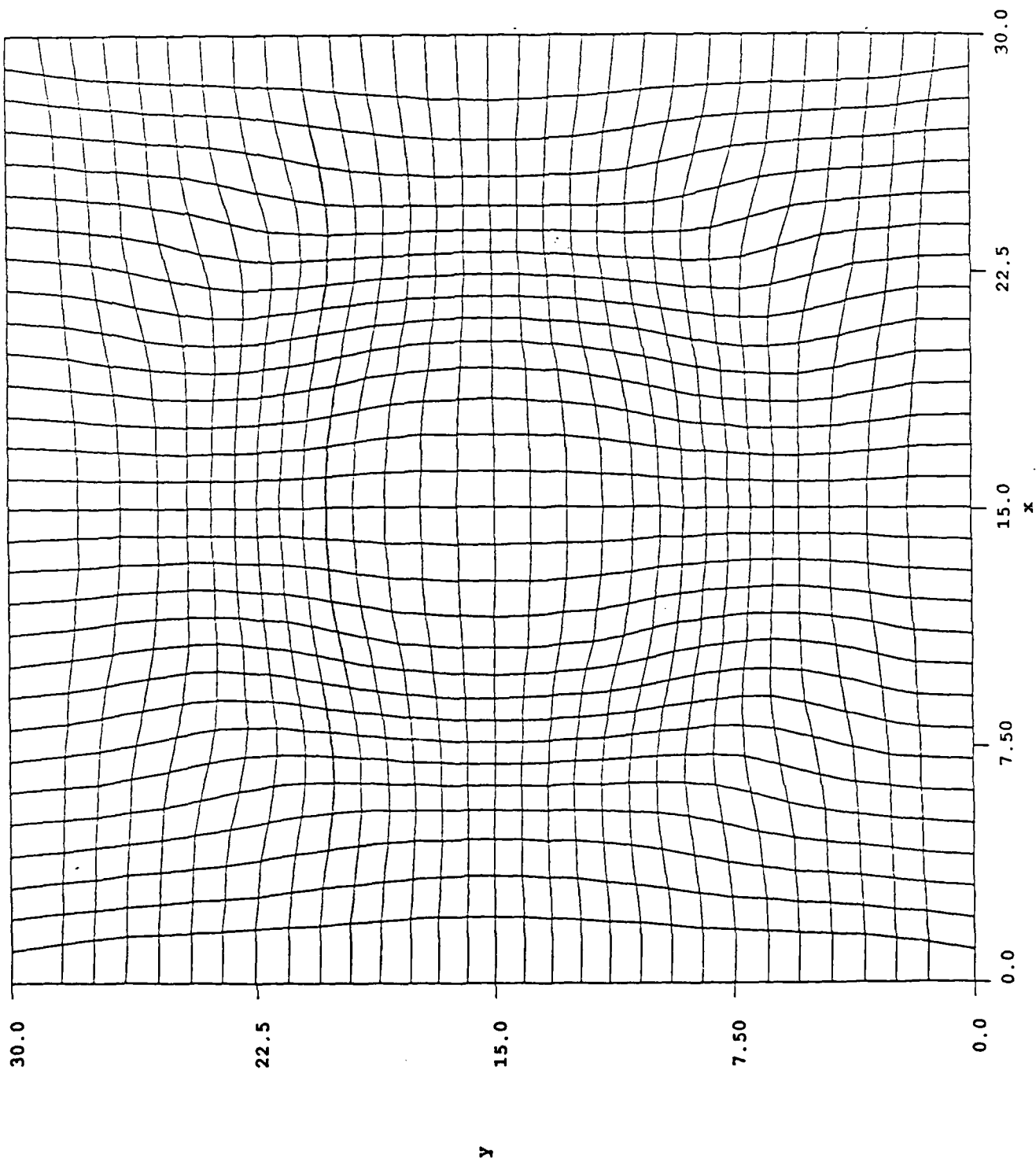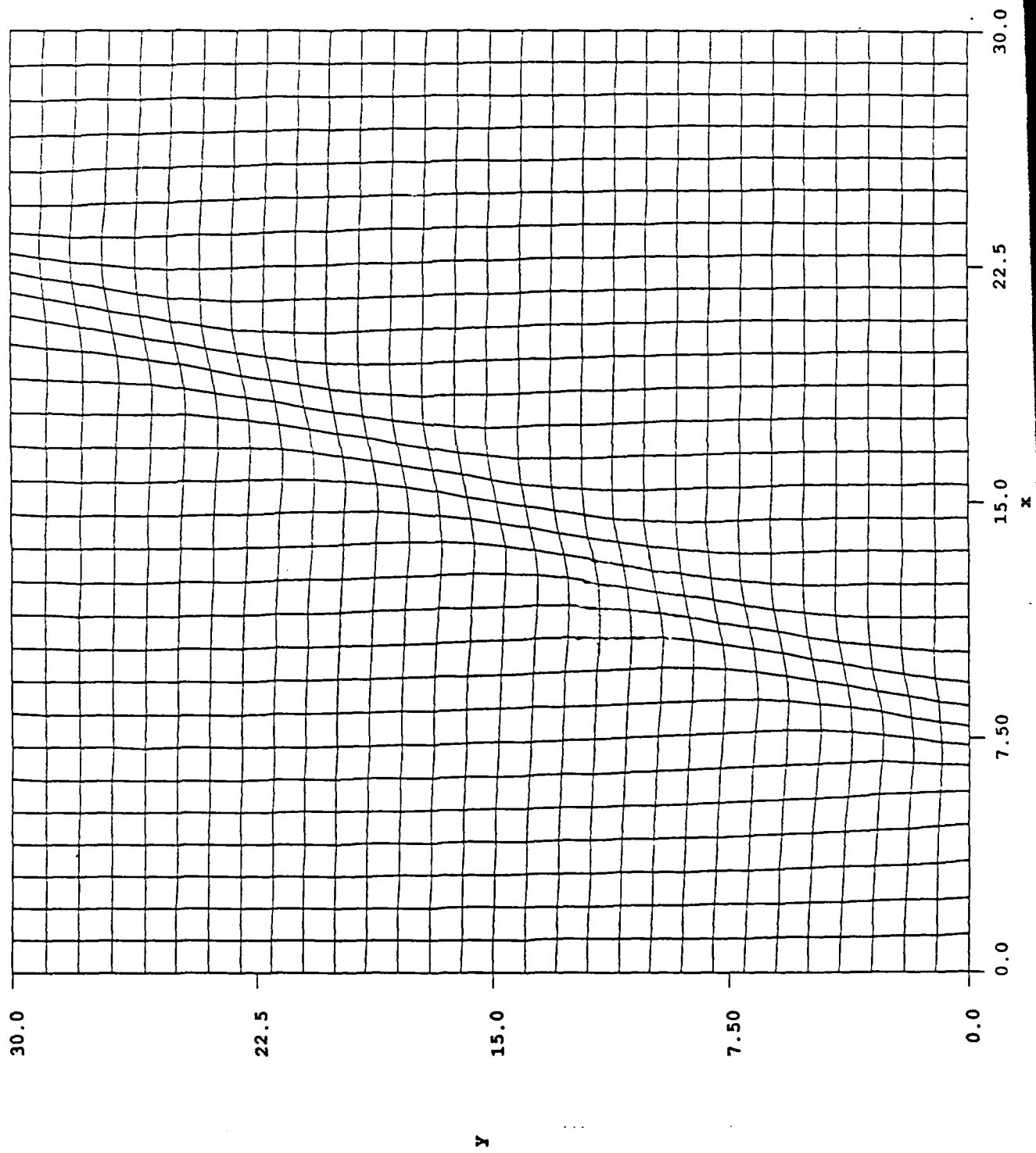Fig. 2 Grid for Shock-like Function

Figure 1

30.0

22.5

15.0

7.50

0.0

y

0.0   7.50   15.0   22.5   30.0

x

Equidistribution Schemes, Poisson Generators, and Adaptive Grids

Dale A. Anderson

Department of Aerospace Engineering
University of Texas
Arlington, Texas 76019

Transmitted by Joe Thompson

ABSTRACT

Equidistribution of a weight function over a mesh is the main concept employed in the recent development of adaptive grid schemes. This idea is reviewed in this paper, and several examples of grids produced using equidistribution are presented. The shortcomings of this approach are identified, and its usefulness is evaluated. Next, the traditional Poisson grid generators are written as nonlinear equidistribution schemes. In this form, the relationship between the numerical solution of a physical problem, the source terms in the grid generation equations, and the weight functions in equidistribution schemes are identified. Examples are given illustrating the ease of providing a useful adaptive grid with this approach.

INTRODUCTION

The idea of constructing an adaptive grid is appealing when systems of partial differential equations are solved using numerical techniques. Conceptually, an adaptive grid can be constructed when any type of partial differential equation is solved. By the very nature of an adaptive grid, intermediate results are required. For hyperbolic or parabolic equations, a marching direction exists and the necessary data are available at the end of each time or timelike step. For elliptic equations, a relaxation scheme can be used to provide the evolving solution. By using the intermediate results, the grid can be adjusted to position points where a need exists to refine the mesh. Normally, grid-point motion is desired to provide better resolution in regions where rapid changes in the solution occur or to provide some reduction in

global error. These two reasons for repositioning grid points are not mutually exclusive, and an improvement in one results in an improvement in the other. The basic idea reduces to providing automatic grid adjustment to resolve details of the solution in regions of varying length scale.

Dwyer et al. [1] have computed solutions for several one-dimensional fluid-dynamics problems including adaptive gridding. Mesh points were positioned in regions where solution gradients were large, using the idea of equidistribution. Rai and Anderson [2, 3] employed an attraction scheme for an adaptive grid based on equidistribution of error on the mesh. They solved a number of problems in both one and two dimensions and illustrated the usefulness of a dynamically adjusting grid in multidimensional applications. Gnoffo [4] also developed an adaptive grid scheme similar to that employed in Reference [1]. Gnoffo successfully applied his technique to compute solutions of a number of complicated fluid-dynamics problems. In one dimension, all of these methods work well. Anderson [5] unified many of these schemes and showed that the idea of equidistribution was common to most of them. A two-dimensional adaptive grid scheme based on equidistribution was also presented in Reference [5] and was used to demonstrate that lack of smoothness and control over grid skewing are limiting factors in the application of such schemes in more than one dimension.

Nakahashi and Deiwert [6] have applied the equidistribution concept in producing solution-adaptive grids. In addition to the usual weight function controlling grid-point spacing, grid skewness control was incorporated through torsion spring coefficients. Results produced and included in Reference [6] represent the successful application of the technique in both two and three dimensions. However, the method does not include any smoothness measure and requires three torsion constants in addition to the weight functions at each point. For these reasons alternative techniques may be desirable.

Brackbill and Saltzman [7] have addressed the difficult multidimensional problem through a variational approach. A functional composed of contributions from an orthogonality condition, a smoothness measure, and adaptivity was minimized, resulting in a system of elliptic partial differential equations to solve for the grid-point locations. Mathematically, this approach is very attractive, and it produces excellent results for physical problems. Practically, the method becomes nearly intractable in three dimensions when the grid is calculated at each iteration level.

It is clear that a new way of producing an adaptive grid which retains some of the properties incorporated in Brackbill and Saltzman's formulation is needed. This scheme should require relatively modest amounts of computer storage and CPU time for practical problems. Grid generation schemes, such as that of Thompson et al. [8], based on solving a system of Poisson equations are most attractive. The governing partial differential equations for the grid

possess all of the requisite properties of smoothness and relatively low skewness. The problem remains to make this method adaptive by relating the source functions to the solution of some physical problem.

In the following sections, the concept of equidistribution along arcs is reviewed, and results of generating grids using strict equidistribution for several examples are presented. The Thompson scheme is then shown to represent a nonlinear equidistribution law when the Poisson grid equations are written along constant computational coordinate surfaces. Finally, the interpretation of Thompson's method as a nonlinear equidistribution law provides a direct relationship between the source terms of the Poisson equations and the solution of the physical problem under consideration. A number of simple examples are given showing the utility of using Thompson's method as an adaptive grid scheme.

EQUIDISTRIBUTION

The equidistribution concept for generating a grid is most easily explained if a single space dimension is considered. In this case, the governing partial differential equation is transformed from physical space (x, t) to a computational domain (ξ, τ), where the numerical solution is calculated. When equidistribution is used to establish the mesh, the spatial locations of the grid points are determined by the equation

$$x_\xi w = \text{constant}, \tag{1}$$

where w is the weight function. With this formulation, the weight function is said to be equidistributed over the mesh. A new grid position in the physical plane is established for each unit change in ξ. This may be seen by assuming that $\Delta\xi$, the spacing in the computational domain, is set at a value of unity. With this assumption, (1) may be approximated by

$$\Delta x w = c = \text{constant}. \tag{2}$$

The physical mesh increment Δx is small when w is large and vice versa.

The grid law provided by (1) can be used to generate a grid by two different methods. Since most researchers are familiar with second-order partial differential equations as grid generators, it is appropriate to write (1) in the form

$$x_{\xi\xi} + x_\xi \frac{w_\xi}{w} = 0. \tag{3}$$

This expression can be solved for the new mesh-point locations using standard finite-difference techniques. Equation (1) can also be solved for either x or ξ using simple numerical quadrature. Let $x = x_{max}$ at $\xi = \xi_{max}$, and integrate to obtain

$$\frac{x}{x_{max}} = \frac{\int_0^\xi \frac{1}{w} \, d\xi}{\int_0^{\xi_{max}} \frac{1}{w} \, d\xi}. \tag{4}$$

It is worthwhile noting that this is the form of the equidistribution law obtained when the physical coordinates are directly computed. The computational coordinate ξ can be computed as an alternative means of finding the grid points. The computational coordinates are given by

$$\frac{\xi}{\xi_{max}} = \frac{\int_0^x w \, dx}{\int_0^{x_{max}} w \, dx}. \tag{5}$$

When the computational coordinates are computed with (5), the physical coordinates are obtained by interpolation.

As noted earlier, the one-dimensional equidistribution law has been used successfully in computing solutions to a number of physical problems. The weight function w is related to the solution and typically is of the form

$$w = 1 + A \left| \frac{h}{h_{max}} \right|, \tag{6}$$

where h represents a gradient or some similar measure of the solution where better resolution is desired, and A is adjusted to alter the adaptivity in the mesh. The unity term is inserted to prevent excessive mesh stretching where the value of h vanishes.

The equidistribution law (1) has also been written for curvilinear arcs in the form

$$S_\xi w_1 = \text{constant}, \tag{7}$$

where S is arc length and w_1 is again a weight function. Equation (7) is

written along a constant-η surface, and a similar expression may be written along a constant-ξ curve,

$$N_\eta w_2 = \text{constant}, \tag{8}$$

where N is arc length and w_2 is a second weight function.

As in the one-dimensional case, (7) and (8) can be solved for (x, y) by simple quadrature or by computing a solution of the governing partial differential equations. If the quadrature approach is selected [see Equation (4)], the changes in arc lengths, ΔS and ΔN, are computed and these are related to changes in x and y through the expression

$$\Delta x = \frac{x_\xi}{S_\xi} \Delta S + \frac{x_\eta}{N_\eta} \Delta N, \tag{9}$$

$$\Delta y = \frac{y_\xi}{S_\xi} \Delta S + \frac{y_\eta}{N_\eta} \Delta N. \tag{10}$$

New values of (x, y) are determined by adding these incremental changes to the original values.

If (7) and (8) are differentiated, the second-order expressions corresponding to (3) are obtained:

$$S_{\xi\xi} + S_\xi \frac{w_{1\xi}}{w_1} = 0, \tag{11}$$

$$N_{\eta\eta} + N_\eta \frac{w_{2\eta}}{w_2} = 0. \tag{12}$$

In these equations

$$S_\xi = \left(x_\xi^2 + y_\xi^2 \right)^{1/2}, \tag{13a}$$

$$N_\eta = \left(x_\eta^2 + y_\eta^2 \right)^{1/2}. \tag{13b}$$

The partial differential equations for (x, y) directly are

$$x_\xi x_{\xi\xi} + y_\xi y_{\xi\xi} + \left(x_\xi^2 + y_\xi^2 \right) \frac{w_{1\xi}}{w_1} = 0, \tag{14}$$

$$x_\eta x_{\eta\eta} + y_\eta y_{\eta\eta} + \left(x_\eta^2 + y_\eta^2 \right) \frac{w_{2\eta}}{w_2} = 0. \tag{15}$$
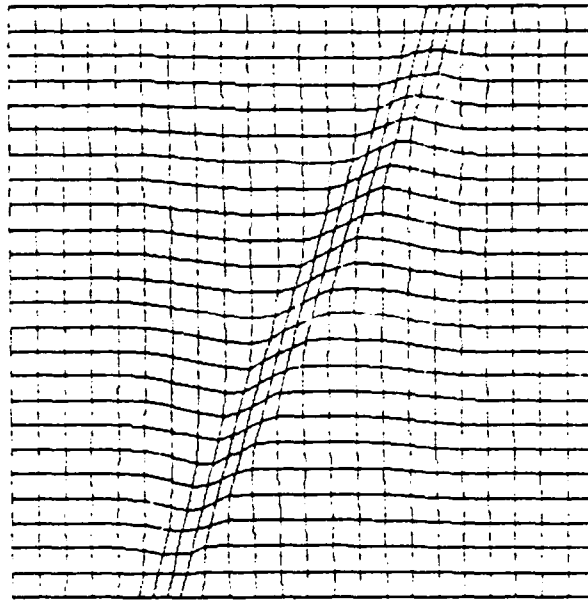
FIG. 1.   Grid adaption to shocklike function, 24×24 grid:

$$u = \begin{cases} 1.0, & 0 \leqslant x \leqslant 0.5(y+10), \\ 0.5(7-x)+0.25y, & 0.5(y+10) \leqslant x \leqslant 0.5(y+14), \\ 0.0, & 0.5(y+14) \leqslant x \leqslant 24. \end{cases}$$

Equations (14) and (15) form an elliptic system which can be solved for (x, y) when the weight functions are prescribed. Typical examples of grids produced with this approach are shown in Figures 1–4. In these calculations, the weight function in both the ξ and η directions is based upon gradients of some prescribed function in that direction. For example, if u is a given function of (x, y), the weight function w_1 is written

$$w_1 = 1 + A\left(\frac{\partial u}{\partial \xi}\right)^2. \tag{16}$$

A similar expression for w_2 is used, incorporating the gradient in the η direction. The value of u is simply a prescribed function of (x, y) as noted in the figure captions. With this approach, a grid can be constructed with an adaptive scheme without the complications of an additional algorithm to compute the solution of some other system of partial differential equations.

Several observations can be made on the basis of the example problems shown. First, the idea of equidistribution can be used to generate grids.
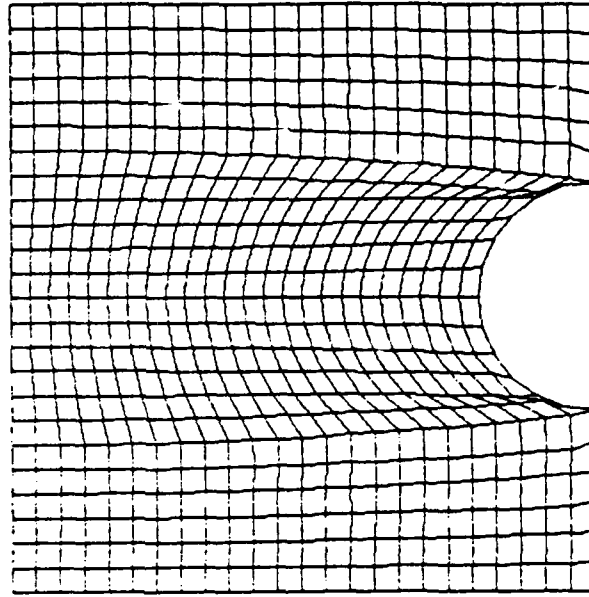
FIG. 2. Grid for circular notch problem, 24 × 24 grid: equidistribution with unity weight function.
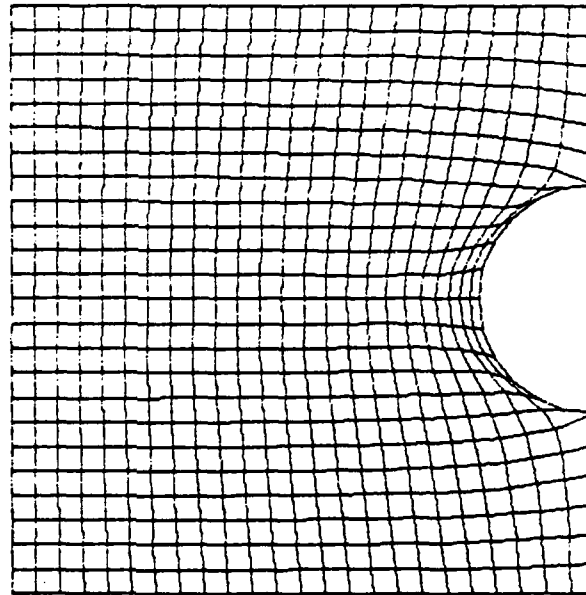

FIG. 3. Grid for circular notch problem: Thompson's scheme, $P, Q = 0$.
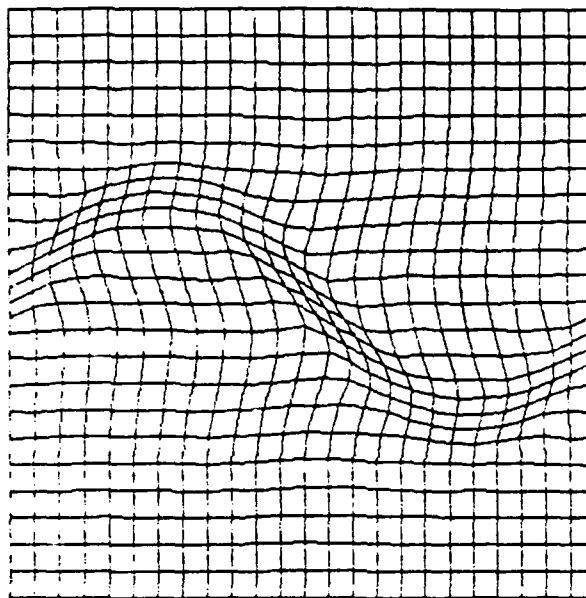
FIG. 4. Equidistribution grid for sine-wave problem:

$$u = \begin{cases} 0, & 0 \leqslant y \leqslant 11 + 4\sin(\pi x/12), \\ 0.5[y - 11 - 4\sin(\pi x/12)], & 11 + 4\sin(\pi x/12) \leqslant y \leqslant 13 + 4\sin(\pi x/12), \\ 1.0, & 13 + 4\sin(\pi x/12) \leqslant y \leqslant 24. \end{cases}$$

However, several properties of this method must be considered. The grid produced using a simple shocklike function shown in Figure 1 is satisfactory. However, some minor skewing occurs even for this problem, where the value of A (ξ direction) was 3 and the value of the adaptation constant in the η direction was unity.

In Figure 2, a grid is presented for a rectangular domain with a circular notch on the boundary. For this example, the weight functions were set equal to one and the grid is not adapted to any prescribed function. The results of using equidistribution to construct a grid with a unit weight function show that any discontinuities in boundary slopes are propagated into the interior grid. For comparison, the grid generated using Laplace's equation

$$\nabla^2 \xi = 0, \tag{17a}$$

$$\nabla^2 \eta = 0 \tag{17b}$$

is shown in Figure (3). This grid is very smooth, but does not provide points in certain regions of the domain. Comparison of the computed results shows

that a specific lack of smoothness exists in the independent arc equidistribution scheme. On the other hand, grid points are uniformly distributed on constant-(ξ, η) curves when equidistribution is used.

The last example presented in this section is that of constructing an adaptive grid on a domain where the adaptation function u goes from 1 to 0 along a sinusoidal curve. From Figure 4, it is clear that significant grid skewness occurs in regions where large adaptation in both directions is requested. Based upon these examples, it does not appear that the use of equidistribution along independent arcs as a multidimensional adaptive grid generator is appropriate (see Reference [5]). Recently, Eiseman [9] has presented results showing better control of skewness by using weight functions based on curvature of the solution surface. However, the basic problems with the equidistribution scheme are not in the choice of weight function but rather are due to shortcomings of the system of governing partial differential equations.

POISSON GRID GENERATORS

One of the most widely used grid generation schemes is based on solving Poisson equations for the grid-point locations [10]. Winslow [11] originally employed Laplace's equation [cf. Equation (17)] to provide the relationship between physical and computational space. This approach provides the smoothest grid possible when the smoothness measure is chosen to be the integral

$$I = \int \int \left[(\nabla \xi)^2 + (\nabla \eta)^2 \right] dx\, dy \tag{18}$$

The Euler-Lagrange equations which result from minimizing this integral are the Laplace grid generation equations [7].

The grid structure produced using (17) is very smooth. A major problem with this scheme is that the interior grid structure depends entirely on boundary data. Consequently, it is difficult to exercise control over the interior point locations. Thompson et al. [8] suggested that the addition of inhomogeneous terms (grid control functions) to (17) would provide a means of interior point control. In two dimensions, the Thompson scheme is written as

$$\nabla^2 \xi = P, \tag{19a}$$

$$\nabla^2 \eta = Q, \tag{19b}$$

where P and Q are selected to position the grid points where desired. These equations are usually solved for the physical coordinates by inverting the role of the dependent and independent variable. These equations are

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = -J^2\left[Px_\xi + Qx_\eta\right], \tag{20a}$$

$$\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = -J^2\left[Py_\xi + Qy_\eta\right], \tag{20b}$$

where

$$\alpha = x_\eta^2 + y_\eta^2, \tag{21a}$$

$$\beta = x_\xi x_\eta + y_\xi y_\eta, \tag{21b}$$

$$\gamma = x_\xi^2 + y_\xi^2, \tag{21c}$$

and the Jacobian of the transformation is given by

$$J = x_\xi y_\eta - x_\eta y_\xi. \tag{22}$$

Grids generated using (20), at least in theory, can be controlled. This is accomplished by appropriate selection of the grid control functions, P and Q. One of the problems with this approach is to select proper values of P and Q on the interior to provide the desired controls.

An interesting attempt at establishing the necessary values of P and Q was presented by Thomas and Middlecoff [12]. In their approach, the boundary point distribution is assumed to be given, and the boundary values of P and Q are calculated from these given data. The grid control functions are then evaluated on the interior of the domain by using an appropriate interpolation procedure. In order to understand the details, it is appropriate to review their scheme.

Let P and Q be written in the form

$$P = \left[\xi_x^2 + \xi_y^2\right]\phi(\xi, \eta), \tag{23a}$$

$$Q = \left[\eta_x^2 + \eta_y^2\right]\psi(\xi, \eta). \tag{23b}$$

With this notation, the grid generation equations become

$$\alpha\left[x_{\xi\xi} + \phi x_{\xi}\right] - 2\beta x_{\xi\eta} + \gamma\left[x_{\eta\eta} + \psi x_{\eta}\right] = 0, \tag{24a}$$

$$\alpha\left[y_{\xi\xi} + \phi y_{\xi}\right] - 2\beta y_{\xi\eta} + \gamma\left[y_{\eta\eta} + \psi y_{\eta}\right] = 0. \tag{24b}$$

The vector form of these equations is frequently used. If

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y,$$

Equations (24a) and (24b) may be written

$$\alpha\left[\mathbf{r}_{\xi\xi} + \phi\mathbf{r}_{\xi}\right] - 2\beta\mathbf{r}_{\xi\eta} + \gamma\left[\mathbf{r}_{\eta\eta} + \psi\mathbf{r}_{\eta}\right] = 0. \tag{25}$$

This expression can be written as a second-order differential equation for arc length along either constant-ξ or -η curves. To obtain the appropriate expression along constant-η lines, eliminate ψ from (25) by taking the cross product with $\mathbf{r}_{\eta}$. This results in the relationship

$$\alpha\left[y_{\eta}\left(x_{\xi\xi} + \phi x_{\xi}\right) - x_{\eta}\left(y_{\xi\xi} + \phi y_{\xi}\right)\right] = y_{\eta}^2\left[2\beta\frac{\partial}{\partial\xi}\left(\frac{x_{\eta}}{y_{\eta}}\right) + \gamma\frac{\partial}{\partial\eta}\left(\frac{x_{\eta}}{y_{\eta}}\right)\right]. \tag{26}$$

Let the local inclination of the constant-ξ curves be denoted by ν (Figure 5). The local slope is defined as

$$\left(\frac{dy}{dx}\right)_{\xi-\text{constant}} = \tan\nu = \frac{y_{\eta}}{x_{\eta}}. \tag{27}$$

Along the constant-ξ curves, the local curvature is

$$k_{\xi} = \frac{\partial\nu}{\partial N} = \frac{\partial\nu}{\partial\eta}\frac{1}{\partial N/\partial\eta}.$$

That is, the curvature for a given line depends only on η and is directly related to $\partial\nu/\partial\eta$. From (27), this function may be written

$$\frac{\partial\nu}{\partial\eta} = \cos^2\nu\,\frac{\partial}{\partial\eta}\left(\frac{y_{\eta}}{x_{\eta}}\right). \tag{28}$$
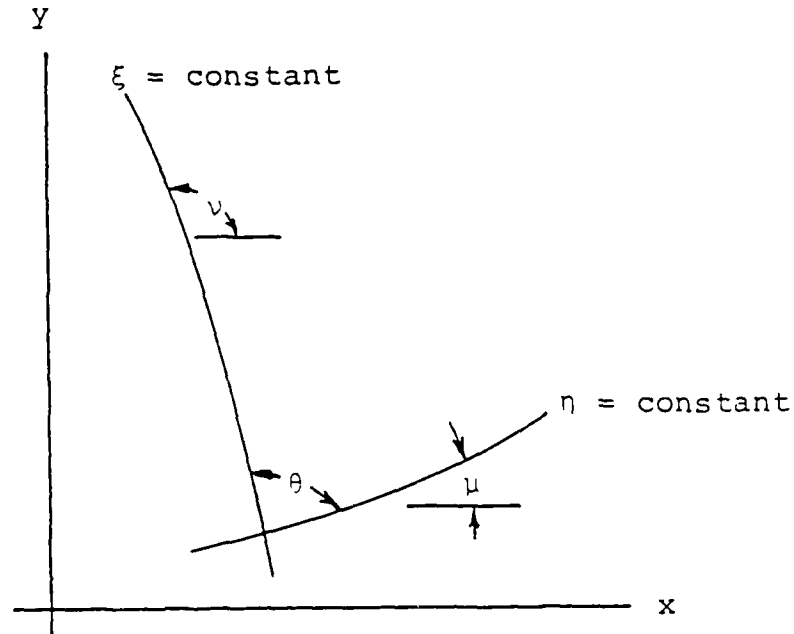
FIG. 5. Notation and grid geometry in physical domain.

If it is assumed that the curvature of the constant-ξ lines is zero as they intersect the constant-η curves, then $\partial v/\partial \eta = 0$. In addition, if a local orthogonality is assumed, the right-hand side of (26) vanishes. Using the arc-length derivative definitions given in (13), the final result may be written

$$S_{\xi\xi} + \dot{\phi}S_{\xi} = 0. \tag{29}$$

Along the constant-ξ curves

$$N_{\eta\eta} + \psi N_{\eta} = 0. \tag{30}$$

On the $\eta = $ constant boundary the grid-point locations are specified. This means that arc lengths are given. If (29) is differenced, the value of ϕ may be determined at every point on the boundary. In a similar manner, the values of ψ on the constant-ξ boundary may be established from (30). Values for ϕ and ψ on the interior can now be determined by using an interpolation scheme.

ADAPTIVE GRIDS

It is interesting to note that the reduced equations (29) and (30) are exactly equidistribution expressions given in differential form as (11) and

(12). By comparing these expressions, the weight functions are related to the grid control functions by the equations

$$\phi = \frac{1}{w_1} \frac{\partial w_1}{\partial \xi},$$ (31a)

$$\psi = \frac{1}{w_2} \frac{\partial w_2}{\partial \eta}$$ (31b)

or

$$w_1 = c_1 e^{\int_\xi \phi \, d\xi},$$ (32a)

$$w_2 = c_2 e^{\int_\eta \psi \, d\eta}.$$ (32b)

As noted previously, the concept of equidistribution as a grid generator has not met with great success and is not recommended. However, the idea is of great value in conceptualizing grid schemes.

Since the reduced equations on constant-ξ and -η surfaces result in equidistribution schemes, it is natural to wonder if the equations on these curves written without assumptions on curvature and orthogonality might look like an equidistribution scheme. Without assumptions on the behavior of the two families of curves at intersections, these equations may be written as

$$S_{\xi\xi} + S_\xi \left[\phi - \left(\mu_\xi - 2\nu_\xi \right) \cot \theta - \frac{S_\xi \nu_\eta}{N_\eta \sin \theta} \right] = 0$$ (33)

and

$$N_{\eta\eta} + N_\eta \left[\psi + \left(\nu_\eta - 2\mu_\eta \right) \cot \theta + \frac{\mu_\xi N_\eta}{S_\xi \sin \theta} \right] = 0.$$ (34)

As noted in Figure 5, θ is the angle of intersection and μ and ν are the slopes of the constant-η and -ξ curves respectively. Equations (33) and (34) may be viewed as nonlinear equidistribution equations. The first term in each of the square brackets is the grid control function and is related to the original P and Q by (23). The second terms are predominately controlled by the intersection angle of the grid lines. If the grid lines intersect at 90°, this term vanishes. However, the $\cot \theta$ is multiplied by curvature terms which also
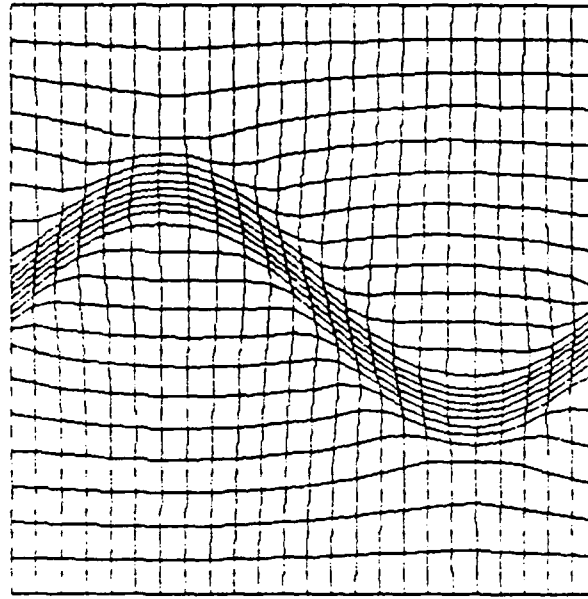
FIG. 6.   Grid for sine-wave problem: P and Q related to $u(x, y)$ (Equation 33).

influence the magnitude of the expression. The last term is primarily controlled by the curvature.

If the last two terms of the bracketed expressions in (33) and (34) are small, the equidistribution law is essentially recovered. Consequently, if ϕ and ψ are defined by (31), the grid becomes adaptive if ϕ and ψ are sufficiently large. In order to test this idea, examples of grids computed using the Thompson scheme with P and Q defined in terms of a weight function are necessary.

The first example selected was the same sine curve as previously used in the equidistribution case. Again, the weight functions were selected to depend on the gradient information [cf. Equation (16)]. The results of using Thompson's scheme for this case are shown in Figure 6. For large values of the clustering parameters, the grid scheme produces a mesh that is adaptive. The adaptation is exactly as desired, following the gradient line as prescribed. The shock-wave grid of Figure 1 was also recalculated using the adaptive scheme based on the Poisson generator, and the results are shown in Figure 7. Again, the adaptation to the change in the prescribed function is clear. As a final example, the solution to a more complicated adaptation problem is shown in Figure 8. In this case, the prescribed function was set equal to zero in the lower right half of the rectangle, and equal to one-half on the upper left half of the domain except for a small semicircular region where the function has a value of unity. The adaptation to the functional changes is remarkable. Notice that difference in the magnitude of the gradient across the diagonal is
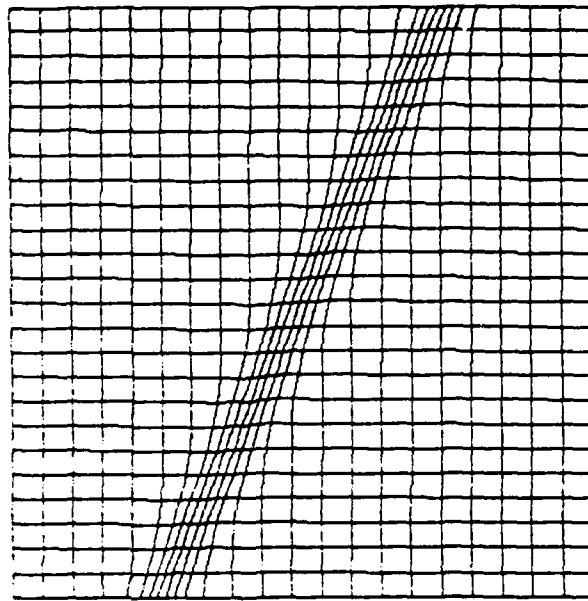
FIG. 7.   Grid for shocklike function: P and Q determined from $u(x, y)$.
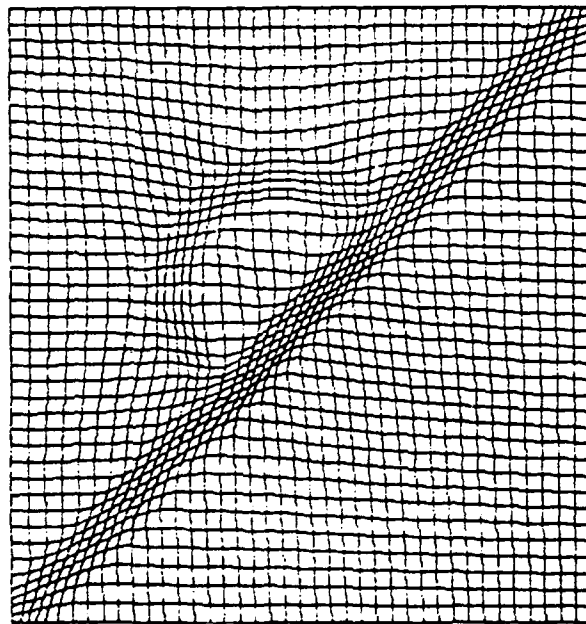

FIG. 8.   Adaptive grid for hat-shaped function.

clearly shown at the boundary of the semicircular region. The circular boundary is also well represented, considering the change from a rectangle to a circular grid.

The Thompson scheme can be used in an adaptive manner as evidenced by the examples calculated. In the solution of a system of partial differential equations, the control functions P and Q are written in terms of some measure of the solution, rather than a prescribed test function as was done for the examples presented here.

The derivation of the relationship between the grid control functions of the Thompson grid scheme and the solution variables for an adaptive scheme was presented for the two-dimensional case. While the three-dimensional development is not included here, the same philosophy leads to a similar identification of the grid control functions with the weight functions of an equidistribution scheme. The lead terms of the Poisson grid scheme written along constant-computational-coordinate surfaces are consistent with those of arc equidistribution.

SUMMARY AND CONCLUSIONS

The idea of equidistribution has been reviewed as a technique for generating grids. The difficulty of using independent arc equidistribution as a grid generator and selecting appropriate weight functions was demonstrated through the use of simple examples. Next, Poisson grid generators were reviewed, emphasizing the difficulty of selecting grid control functions. The Thompson grid generation equations were written along constant-computational-coordinate surfaces and shown to be equivalent to a set of nonlinear equidistribution equations. From this point of view, appropriate choices for the grid control functions on the interior were identified, making the standard grid scheme completely adaptive. This behavior was demonstrated with simple numerical examples.

The concept of using the Poisson grid generators adaptively is significant. The necessary smoothness properties are part of the grid generator (not the weight function), and simple examples illustrate the success of the approach. From an economic view, most organizations do use Poisson-type equations as grid generators, and these can be made into solution-adaptive schemes with near-trivial changes. Consequently, existing software can be retained and no major investment in new programs is necessary to obtain the ability to produce adaptive grids.

REFERENCES

1 H. A. Dwyer, M. O. Smooke, and R. J. Kee, Adaptive grid method for problems in fluid mechanics and heat transfer, *AIAA J.* 18:1205–1212 (1980).

2 M. M. Rai and D. A. Anderson, Grid evolution in time asymptotic problems, *J. Comput. Phys.* 43:327–344 (1981).

3 M. M. Rai and D. A. Anderson, Application of adaptive grids to fluid-flow problems with asymptotic solutions, *AIAA J.* 20:496–502 (1982).

4 P. A. Gnoffo, A Vectorized, Finite-Volume, Adaptive Grid Algorithm Applied to Planetary Entry Problems, AIAA paper 82-1018.

5 D. A. Anderson, Adaptive grid methods for partial differential equations, in *Advances in Grid Generation*, ASME, 1983, pp. 1–15.

6 K. Nakahaski and G. S. Deiwert, Three-dimensional adaptive grid method, *AIAA J.* 24:945–954 (1986).

7 J. Brackbill and J. Saltzman, Adaptive zoning for singular problems in two dimensions, *J. Comput. Phys.* 46:342–368 (1982).

8 J. F. Thompson, F. C. Thames, and C. W. Mastin, Automatic numerical generation of body-fitted curvilinear coordinate system for fields containing any number of arbitrary two-dimensional bodies, *J. Comput. Phys.* 15:299–319 (1974).

9 P. Eiseman, Alternating Direction Adaptive Grid Generation, AIAA Paper 83-1937.

10 J. F. Thompson, Z. U. A. Warsi, and C. W. Mastin, *Numerical Grid Generation*, North-Holland, 1985.

11 A. Winslow, Numerical solution of the quasi-linear Poisson equation, *J. Comput. Phys.* 1:149–172 (1966).

12 P. D. Thomas and J. F. Middlecoff, Direct control of the grid point distribution in meshes generated by elliptic equations, *AIAA J.* 18:652–656 (1980).